

Twisting on associative algebras and Rota-Baxter type operators

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Abstract

We will introduce an operation “twisting” on Hochschild complex by analogy with Drinfeld’s twisting operations. By using the twisting and derived bracket construction, we will study differential graded Lie algebra structures associated with bi-graded Hochschild complex. We will show that Rota-Baxter type operators are solutions of Maurer-Cartan equations. As an application of twisting, we will give a construction of associative Nijenhuis operators.

1 Introduction.

In [7], Drinfeld introduced an operation “twisting”, motivated by the study of quasi-Lie bialgebras and quasi-Hopf algebras. The twisting operations provide a method of analyzing Manin triples. In the context of Poisson geometry, they gave the detailed study of twisting operations (see Kosmann-Schwarzbach [12, 14] and Roytenberg [21, 22]). We shortly describe twisting operations. We consider a graded commutative algebra, $\bigwedge(V \oplus V^*)$, where V is a vector space over \mathbb{R} , V^* is the dual space of V . The graded algebra has a graded Poisson bracket defined by $\{V, V\} = \{V^*, V^*\} := 0$ and $\{V, V^*\} := \langle V, V^* \rangle$. By definition, a *structure* in the graded Poisson algebra is an element Θ in $\bigwedge^3(V \oplus V^*)$ satisfying a Maurer-Cartan equation $\{\Theta, \Theta\} = 0$. It is known that the structure Θ is an invariant Lie algebra structure on $V \oplus V^*$. The structures are closely related with (quasi-)Lie bialgebra structures. A Lie bialgebra structure is defined as a pair of tensors (ν_1, ν_2) such that $\Theta_{12} := \nu_1 + \nu_2$ is a structure in above sense, where $\nu_1 \in (\bigwedge^2 V^*) \otimes V$ and $\nu_2 \in V^* \otimes \bigwedge^2 V$. When (ν_1, ν_2) is a structure of Lie bialgebra, the total space $(V \oplus V^*, \Theta_{12})$ is called a Drinfeld double. Let r be an element in $V \wedge V$. By

definition, the twisting of a structure Θ by r is a canonical transformation;

$$\Theta^r := \exp(X_r)(\Theta),$$

where X_r is a Hamiltonian vector field $X_r := \{-, r\}$ and Θ^r is the result of twisting. Several interesting information is riding on the orbits of twisting operations. We recall a basic proposition. Let $(\nu_1, 0)$ be a structure of Lie bialgebra such that $\nu_2 = 0$. Then the Drinfeld double is the space $V \oplus V^*$ with the structure $\Theta_1 := \nu_1$. If r is a solution of a Maurer-Cartan equation (or classical Yang-Baxter equation)

$$[r, r] = 0,$$

then a pair $(\nu_1, \{\nu_1, r\})$ is a Lie bialgebra structure and the *double*, $\nu_1 + \{\nu_1, r\}$, is equal with the result of twisting Θ_1^r , where $[r, r] := \{\{\nu_1, r\}, r\}$. Conversely, the Maurer-Cartan condition of r is characterized by this proposition.

The aim of this note is to construct the theory of twisting on associative algebras along the philosophy and construction in [14] and [21]. At first, we will define a twisting operation in the category of associative algebras. The twisting operation is defined by using only a canonical bigraded system of the graded Poisson algebra $\bigwedge^*(V \oplus V^*)$. Hence, given a suitable bigraded Lie algebra, one can define a twisting like operation on the bigraded Lie algebra. We consider a Hochschild complex $C^*(\mathcal{T}) := \text{Hom}(\mathcal{T}^{\otimes*}, \mathcal{T})$, where \mathcal{T} is a vector space decomposed into two subspaces $\mathcal{T} := \mathcal{A}_1 \oplus \mathcal{A}_2$. In Section 2, we will introduce a canonical bigraded Lie algebra system on $C^*(\mathcal{A}_1 \oplus \mathcal{A}_2)$. The graded Lie bracket is given by Gerstenhaber's bracket product. Our *structures*, θ , are defined as associative structures on $\mathcal{A}_1 \oplus \mathcal{A}_2$, i.e., θ is a 2-cochain in $C^2(\mathcal{A}_1 \oplus \mathcal{A}_2)$ and $t_1 * t_2 := \theta(t_1 \otimes t_2)$ is associative for any $t_1, t_2 \in \mathcal{A}_1 \oplus \mathcal{A}_2$. For a given 1-cochain $H : \mathcal{A}_2 \rightarrow \mathcal{A}_1$, we define a twisting operation by the same manner with classical one,

$$\theta^H := \exp(X_{\hat{H}})(\theta),$$

where \hat{H} is the image of the natural map $C^*(\mathcal{A}_2, \mathcal{A}_1) \hookrightarrow C^*(\mathcal{A}_1 \oplus \mathcal{A}_2)$ and $X_{\hat{H}}$ is an analogy of Hamiltonian vector field defined by $X_{\hat{H}} := \{-, \hat{H}\}$, where $C^*(\mathcal{A}_2, \mathcal{A}_1) := \text{Hom}(\mathcal{A}_2^{\otimes*}, \mathcal{A}_1)$. We will see that θ is decomposed into the unique 4 substructures,

$$\theta = \hat{\phi}_1 + \hat{\mu}_1 + \hat{\mu}_2 + \hat{\phi}_2.$$

The twisting operation is completely determined by transformation rules of the 4 substructures. In Section 4, we will give explicit formulas of the transformation rules (Theorem 4.5).

We consider the case of $\hat{\phi}_1 = \hat{\phi}_2 = 0$. In this case, \mathcal{A}_1 and \mathcal{A}_2 are both subalgebras of the associative algebra $(\mathcal{A}_1 \oplus \mathcal{A}_2, \theta)$. Such a triple $(\mathcal{A}_1 \oplus \mathcal{A}_2, \mathcal{A}_1, \mathcal{A}_2)$ is called an associative twilled algebra, simply, twilled algebra (Carinena and coauthors [5]). When a Lie algebra is decomposed into two subalgebras, it is called a *twilled* Lie algebra ([13]), or called a twilled extension in [10], or a *double* Lie algebra in [17]. This concept is used in order to construct integrable Hamiltonian systems (Adler-Kostant-Symes

theorem). The notion of associative twilled algebra is considered as an associative version of the classical one. In [5], they studied associative twilled algebras from the point of view of quantization. In Section 3, we will give the detailed study for twilled algebras. By derived bracket construction in [13], a twilled algebra structure on $\mathcal{A}_1 \oplus \mathcal{A}_2$ induces a differential graded Lie algebra (shortly, dg-Lie algebra) structure on $C^*(\mathcal{A}_2, \mathcal{A}_1)$ (see Proposition 3.3). So we can consider a deformation theory on the induced dg-Lie algebra. We consider a Maurer-Cartan equation in the dg-Lie algebra,

$$dR + \frac{1}{2}[R, R] = 0.$$

We can find a solution R in Rota-Baxter algebra theory. Let (\mathcal{A}, R) be an arbitrary associative algebra equipped with an operator $R : \mathcal{A} \rightarrow \mathcal{A}$. The operator R is called a Rota-Baxter operator, if R satisfies an identity (so-called Rota-Baxter identity),

$$R(x)R(y) = R(R(x)y + xR(y)) + qR(xy),$$

where $q \in \mathbb{K}$ is a scalar (called a weight). Rota-Baxter operators have been studied in combinatorics (see Rota [18, 19]). In this note we do not study the combinatorial problem, because it is beyond our aim. Now $\mathcal{A} \oplus \mathcal{A}$ has a natural twilled algebra structure, and then $C^*(\mathcal{A}, \mathcal{A})$ has a dg-Lie algebra structure. In Section 5.1, we will show that R is a Rota-Baxter operator if and only if R is a solution of the Maurer-Cartan equation.

In Section 6, we will give an application of our construction. We recall the notion of associative Nijenhuis operator ([5]). Let $N : \mathcal{A} \rightarrow \mathcal{A}$ be a linear map on an associative algebra \mathcal{A} . The operator N is called an *associative Nijenhuis operator*, if it satisfies an associative version of classical Nijenhuis condition,

$$N(x)N(y) = N(N(x)y + xN(y)) - N^2(xy),$$

where $x, y \in \mathcal{A}$. They showed that a deformed multiplication, $x \times_N y := N(x)y + xN(y) - N(xy)$, is a new associative multiplication on \mathcal{A} and it is compatible with original one. In this sense, an associative Nijenhuis operator induces a quantum bihamiltonian system (see [5]). We will give a construction of associative Nijenhuis operators by analogy with Poisson-Nijenhuis geometry.

We recall a theorem of Vaisman [25]. Let (V, P) be a Poisson manifold equipped with a Poisson structure tensor P , i.e., P is a solution of a Maurer-Cartan equation,

$$\frac{1}{2}[P, P] = 0,$$

where the bracket product is a graded Lie bracket (called Schouten-Nijenhuis bracket). Since the Poisson structure is a $(2, 0)$ -tensor, it is identified with a bundle map $P : T^*V \rightarrow TV$. The Poisson bundle map induces a Lie algebroid structure on the cotangent bundle T^*V , i.e., the space of sections of $\bigwedge^1 T^*V$ has a certain graded Lie bracket $\{\cdot, \cdot\}_P$. He showed that if a 2-form ω is a solution of the strong Maurer-Cartan equation, $d\omega = \{\omega, \omega\}_P = 0$, then the bundle map $N := P\omega : TV \rightarrow TV$ is a Nijenhuis tensor and the pair (P, N) is a compatible pair, or a Poisson-Nijenhuis structure in the sense of [11]. This compatibility implies that the bundle map $NP : T^*V \rightarrow TV$ is a Poisson structure bundle map and $P + tNP$ is a one parameter family of

Poisson structures.

We will show a similar theorem to Vaisman's theorem. First of all, we need Rota-Baxter *type* operators as substitutes for Poisson structures. Let \mathcal{A} be an associative algebra and M an \mathcal{A} -bimodule, and let $\pi : M \rightarrow \mathcal{A}$ be a linear map. The linear map π is called a *generalized Rota-Baxter operator* of weight 0, or shortly GRB ([23]), if π is a solution of

$$\pi(m)\pi(n) = \pi(\pi(m) \cdot n + m \cdot \pi(n)), \quad (GRB)$$

where $m, n \in M$ and \cdot is the bimodule action. When $M = \mathcal{A}$ as a canonical bimodule, (GRB) reduces to a classical Rota-Baxter identity of weight zero. We consider a semidirect product algebra $(\mathcal{T} := \mathcal{A} \ltimes M, \hat{\mu})$, where $\hat{\mu}$ is the associative multiplication of $\mathcal{A} \ltimes M$. The Hochschild complex $C^*(\mathcal{A} \ltimes M)$ becomes a dg-Lie algebra by Gerstenhaber bracket and the coboundary map $d_{\hat{\mu}} := \{\hat{\mu}, -\}$. We define, due to [13], a second bracket product on $C^*(\mathcal{A} \ltimes M)$ by

$$[f, g]_{\hat{\mu}} := (-1)^{|f|-1} \{\{\hat{\mu}, f\}, g\}.$$

Here the new bracket is a graded Lie bracket on $C^*(M, \mathcal{A}) \subset C^*(\mathcal{A} \ltimes M)$. One can show that π is a generalized Rota-Baxter operator if and only if it is a solution of the Maurer-Cartan equation

$$\frac{1}{2}[\hat{\pi}, \hat{\pi}]_{\hat{\mu}} = 0,$$

where $\hat{\pi}$ is the image of the natural map $C^1(M, \mathcal{A}) \hookrightarrow C^1(\mathcal{A} \ltimes M)$, $\pi \mapsto \hat{\pi}$.

Now, given a generalized Rota-Baxter operator $\pi : M \rightarrow \mathcal{A}$, M becomes an associative algebra, where the associative multiplication on M is given by a structure $\{\hat{\mu}, \hat{\pi}\}$. The associativity of $\{\hat{\mu}, \hat{\pi}\}$ is followed from $[\hat{\pi}, \hat{\pi}]_{\hat{\mu}} = 0$. We denote the associative algebra by M_{π} . One can show that $M_{\pi} \oplus \mathcal{A}$ has a twilled algebra structure. Thus a dg-Lie algebra structure, $(d_{\hat{\mu}}, [.,.]_{\{\hat{\mu}, \hat{\pi}\}})$, is induced on $C^*(\mathcal{A}, M_{\pi})$. By analogy with Vaisman's theorem, we assume that $\Omega : \mathcal{A} \rightarrow M$ is a solution of the strong Maurer-Cartan equation in $C^*(\mathcal{A}, M_{\pi})$,

$$d_{\hat{\mu}}\hat{\Omega} = [\hat{\Omega}, \hat{\Omega}]_{\{\hat{\mu}, \hat{\pi}\}} = 0,$$

where $d_{\hat{\mu}}$ is the Hochschild coboundary on $C^*(\mathcal{A}, M)$ and $\hat{\Omega}$ is defined by the similar manner with $\hat{\pi}$. Then we can show that a linear endomorphism $N := \pi\Omega : \mathcal{A} \rightarrow \mathcal{A}$ is an associative Nijenhuis operator and the pair $(\pi, N = \pi\Omega)$ is compatible (see Proposition 6.1). This proposition can be considered as an associative version of Vaisman's result.

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2 Cochain calculus.

In this section, we will define a bigraded Lie algebra structure on Hochschild complex $C^*(\mathcal{A}_1 \oplus \mathcal{A}_2)$. In the following, we assume that the characteristic of a ground field \mathbb{K} is zero and that \mathbb{Q} is included in \mathbb{K} .

2.1 Gerstenhaber brackets.

We recall Gerstenhaber's bracket product. Let V be a vector space over \mathbb{K} . We consider the space of cochains $\mathfrak{g}(V) := \bigoplus_{n \in \mathbb{N}} C^n(V)$, where $C^n(V) = C^n(V, V) := \text{Hom}_{\mathbb{K}}(V^{\otimes n}, V)$. By definition, the degree of $f \in \mathfrak{g}(V)$ is $|f|$, if f is in $C^{|f|}(V)$. For any $f \in C^{|f|}(V)$ and $g \in C^{|g|}(V)$, we define a product,

$$f \bar{\circ} g := \sum_{i=1}^{|f|} (-1)^{(i-1)(|g|-1)} f \circ_i g,$$

where \circ_i is the composition of maps defined by

$$f \circ_i g(b_1, \dots, b_{|f|+|g|-1}) = f(b_1, \dots, b_{i-1}, g(b_i, \dots, b_{i+|g|-1}), b_{i+|g|}, \dots, b_{|f|+|g|-1}).$$

The degree of $f \bar{\circ} g$ is $|f| + |g| - 1$. The Gerstenhaber bracket, or shortly, G-bracket on $\mathfrak{g}(V)$ is defined as a graded commutator,

$$\{f, g\} := f \bar{\circ} g - (-1)^{(|f|-1)(|g|-1)} g \bar{\circ} f.$$

We recall two fundamental identities:

(I) graded commutativity,

$$\{f, g\} = -(-1)^{(|f|-1)(|g|-1)} \{g, f\},$$

(II) graded Jacobi identity,

$$\begin{aligned} (-1)^{(|f|-1)(|h|-1)} \{\{f, g\}, h\} + (-1)^{(|h|-1)(|g|-1)} \{\{h, f\}, g\} + \\ (-1)^{(|g|-1)(|f|-1)} \{\{g, h\}, f\} = 0, \end{aligned}$$

where $h \in C^{|h|}(V)$. The above graded Jacobi identity is equivalent with

$$\{f, \{g, h\}\} = \{\{f, g\}, h\} + (-1)^{(|f|-1)(|g|-1)} \{g, \{f, h\}\}. \quad (II')$$

(II') is called a graded Leibniz identity, or sometimes called, a graded Loday identity.

Graded Lie algebras. Let \mathfrak{g} be a graded vector space equipped with a binary multiplication $\{, \}$ of degree 0. When the bracket product satisfies the following two conditions (1) and (2), \mathfrak{g} is called a graded Lie algebra.

$$\{f, g\} = -(-1)^{\deg(f)\deg(g)} \{g, f\}, \quad (1)$$

$$\{f, \{g, h\}\} = \{\{f, g\}, h\} + (-1)^{\deg(f)\deg(g)} \{g, \{f, h\}\}, \quad (2)$$

where $f, g, h, \in \mathfrak{g}$ and $\deg(-)$ is the degree of \mathfrak{g} . The cochain complex $\mathfrak{g}(V)$ is a graded Lie algebra of $\deg(f) := |f| - 1$. A graded Lie algebra \mathfrak{g} is called a differential graded Lie algebra (dg-Lie algebra), if \mathfrak{g} has a square zero derivation d of degree +1 satisfying,

$$d\{f, g\} = \{df, g\} + (-1)^{\deg(f)} \{f, dg\}. \quad (3)$$

Associative structures. It is well-known that $S \in C^2(V)$ is an associative structure if and only if it is a solution of Maurer-Cartan equation, $\{S, S\} = 0$. If S is an associative structure, then $d_S(f) := \{S, f\}$ is a coboundary map

of Hochschild complex $(C^*(V), d_S)$, and then $(\mathfrak{g}(V), d_S)$ becomes a dg-Lie algebra.

Derived brackets. Let \mathfrak{g} be a dg-Lie algebra. We define a new bracket product by

$$[f, g]_d := (-1)^{\deg(f)} \{df, g\}.$$

The new bracket is called a *derived bracket* ([13]). It is well-known that the derived bracket is a graded Leibniz bracket, i.e., (2) holds up to degree shift. Remark that the derived bracket is not graded commutative in general. We recall a basic lemma.

Lemma 2.1. ([13]) *Let \mathfrak{g} be a dg-Lie algebra, and let $\mathfrak{h} \subset \mathfrak{g}$ be an abelian subalgebra, i.e., $\{\mathfrak{h}, \mathfrak{h}\} = 0$. We define a new degree (derived degree) by $\deg_d(h) := \deg(h) + 1$. If the derived bracket is closed on \mathfrak{h} , then $(\mathfrak{h}, \deg_d, [\cdot, \cdot]_d)$ is a graded Lie algebra.*

2.2 Lift and Bidegree.

Let \mathcal{A}_1 and \mathcal{A}_2 be vector spaces, and let $c : \mathcal{A}_2^{\otimes n} \rightarrow \mathcal{A}_1$ be a linear map, or a cochain in $C^n(\mathcal{A}_2, \mathcal{A}_1)$. We can construct a cochain $\hat{c} \in C^n(\mathcal{A}_1 \oplus \mathcal{A}_2)$ by

$$\hat{c}\left((a_1, x_1) \otimes \dots \otimes (a_n, x_n)\right) := (c(x_1, \dots, x_n), 0).$$

In general, for a given multilinear map $f : \mathcal{A}_{i(1)} \otimes \mathcal{A}_{i(2)} \otimes \dots \otimes \mathcal{A}_{i(n)} \rightarrow \mathcal{A}_j$, $i(1), \dots, i(n), j \in \{1, 2\}$, we define a cochain $\hat{f} \in C^n(\mathcal{A}_1 \oplus \mathcal{A}_2)$ by

$$\hat{f} := \begin{cases} f & \text{on } \mathcal{A}_{i(1)} \otimes \mathcal{A}_{i(2)} \otimes \dots \otimes \mathcal{A}_{i(n)}, \\ 0 & \text{all other cases.} \end{cases}$$

We call the cochain \hat{f} a horizontal lift of f , or simply, lift. For instance, the lifts of $\alpha : \mathcal{A}_1 \otimes \mathcal{A}_1 \rightarrow \mathcal{A}_1$, $\beta : \mathcal{A}_1 \otimes \mathcal{A}_2 \rightarrow \mathcal{A}_2$ and $\gamma : \mathcal{A}_2 \otimes \mathcal{A}_1 \rightarrow \mathcal{A}_2$ are defined by, respectively,

$$\hat{\alpha}((a, x), (b, y)) = (\alpha(a, b), 0), \quad (4)$$

$$\hat{\beta}((a, x), (b, y)) = (0, \beta(a, y)), \quad (5)$$

$$\hat{\gamma}((a, x), (b, y)) = (0, \gamma(x, b)). \quad (6)$$

Let $H : \mathcal{A}_2 \rightarrow \mathcal{A}_1$ (resp. $H : \mathcal{A}_1 \rightarrow \mathcal{A}_2$) be a 1 cochain. The lift is defined by

$$\hat{H}(a, x) = (H(x), 0) \quad (\text{resp. } \hat{H}(a, x) = (0, H(a))).$$

For any $(a, x) \in \mathcal{A}_1 \oplus \mathcal{A}_2$, we have $\hat{H}\hat{H}(a, x) = \hat{H}(H(x), 0) = (0, 0)$.

Lemma 2.2. $\hat{H}\hat{H} = 0$.

This lemma will be used in Section 4.

We denote by $\mathcal{A}^{l,k}$ the direct sum of all $l + k$ -tensor powers of \mathcal{A}_1 and \mathcal{A}_2 , where l (resp. k) is the number of \mathcal{A}_1 (resp. \mathcal{A}_2). For instance,

$$\mathcal{A}^{1,2} := (\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \mathcal{A}_2) \oplus (\mathcal{A}_2 \otimes \mathcal{A}_1 \otimes \mathcal{A}_2) \oplus (\mathcal{A}_2 \otimes \mathcal{A}_2 \otimes \mathcal{A}_1).$$

The tensor space $(\mathcal{A}_1 \oplus \mathcal{A}_2)^{\otimes n}$ is expanded into the direct sum of $\mathcal{A}^{l,k}$, $l + k = n$. For instance,

$$(\mathcal{A}_1 \oplus \mathcal{A}_2)^{\otimes 2} = \mathcal{A}^{2,0} \oplus \mathcal{A}^{1,1} \oplus \mathcal{A}^{0,2}.$$

We consider the space of cochains, $C^n(\mathcal{A}_1 \oplus \mathcal{A}_2) := \text{Hom}_{\mathbb{K}}((\mathcal{A}_1 \oplus \mathcal{A}_2)^{\otimes n}, \mathcal{A}_1 \oplus \mathcal{A}_2)$. By the standard properties of Hom-functor, we have

$$C^n(\mathcal{A}_1 \oplus \mathcal{A}_2) \cong \sum_{l+k=n} C^n(\mathcal{A}^{l,k}, \mathcal{A}_1) \oplus \sum_{l+k=n} C^n(\mathcal{A}^{l,k}, \mathcal{A}_2), \quad (7)$$

where the isomorphism is the horizontal lift.

Let f be a n -cochain in $C^n(\mathcal{A}_1 \oplus \mathcal{A}_2)$. We say the *bidegree* of f is $k|l$, if f is an element in $C^n(\mathcal{A}^{l,k-1}, \mathcal{A}_1)$ or in $C^n(\mathcal{A}^{l-1,k}, \mathcal{A}_2)$, where $n = l + k - 1$. We denote the bidegree of f by $\|f\| = k|l$. In general, cochains do not have bidegree. We call a cochain f a *homogeneous cochain*, if f has the bidegree.

We have $k + l \geq 2$, because $n \geq 1$. Thus there are no cochains of bidegree $0|0$ or $1|0$ or $0|1$. If the dimension of \mathcal{A}_1 is finite and $\mathcal{A}_2 = \mathcal{A}_1^*$ is the dual space of \mathcal{A}_1 , then a $k|l$ -cochain is identified with an element in $\mathcal{A}_1^{\otimes k} \otimes \mathcal{A}_1^{*\otimes l}$. Hence the definition above is compatible with the classical one. For instance, the lift of $H : \mathcal{A}_2 \rightarrow \mathcal{A}_1$, $\hat{H} \in C^1(\mathcal{A}_1 \oplus \mathcal{A}_2)$, has the bidegree $2|0$. We recall $\hat{\alpha}, \hat{\beta}, \hat{\gamma} \in C^2(\mathcal{A}_1 \oplus \mathcal{A}_2)$ in (4), (5) and (6). One can easily see $\|\hat{\alpha}\| = \|\hat{\beta}\| = \|\hat{\gamma}\| = 1|2$. Thus the sum of $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$,

$$\hat{\mu} := \hat{\alpha} + \hat{\beta} + \hat{\gamma} \quad (8)$$

is a homogeneous cochain with bidegree $1|2$. The cochain $\hat{\mu}$ is a multiplication of semidirect product type,

$$\hat{\mu}((a, x), (b, y)) = (\alpha(a, b), \beta(a, y) + \gamma(x, b)),$$

where $(a, x), (b, y) \in \mathcal{T}$. Remark that $\hat{\mu}$ is not lift (there is no μ), however, we will use this symbol, because $\hat{\mu}$ is an interesting homogeneous cochain.

Clearly, the lemma below holds.

Lemma 2.3. *Let $f \in C^n(\mathcal{A}_1 \oplus \mathcal{A}_2)$ be a cochain. The bidegree of f is $k|l$ if and only if the following 4 conditions hold.*

(deg1) $k + l - 1 = n$.

(deg2-1) If \mathbf{x} is an element in $\mathcal{A}^{l,k-1}$, then $f(\mathbf{x})$ is in \mathcal{A}_1 .

(deg2-2) If \mathbf{x} is an element in $\mathcal{A}^{l-1,k}$, then $f(\mathbf{x})$ is in \mathcal{A}_2 .

(deg3) All the other cases, $f(\mathbf{x}) = 0$.

Lemma 2.4. *If $\|f\| = k|0$ (resp. $0|k$) and $\|g\| = l|0$ (resp. $0|l$), then $\{f, g\} = 0$, or simply,*

$$\{(k|0), (l|0)\} = \{(0|k), (0|l)\} = 0.$$

Proof. Assume that $\|f\| = k|0$ and $\|g\| = l|0$. Then f and g are both horizontal lifts of cochains in $C^*(\mathcal{A}_2, \mathcal{A}_1)$. Thus, from the definition of lift, we have $f \circ_i g = g \circ_j f = 0$ for any i, j . \square

Lemma 2.5. *Let $f \in C^{|f|}(\mathcal{A}_1 \oplus \mathcal{A}_2)$ and $g \in C^{|g|}(\mathcal{A}_1 \oplus \mathcal{A}_2)$ homogeneous cochains with bidegrees $k_f|l_f$ and $k_g|l_g$, respectively, where $|f|$ and $|g|$ are usual degrees of cochains f and g . The composition $f \circ_i g$ is again a homogeneous cochain, and the bidegree is $k_f + k_g - 1|l_f + l_g - 1$.*

Proof. We show the conditions (deg1)-(deg3). The condition (deg1) holds, because $k_f + k_g - 1 + l_f + l_g - 1 = |f| + |g| = |f \circ_i g| + 1$. We show the condition (deg2). Take an element $\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}$ in $\mathcal{A}^{l_f + l_g - 1, k_f + k_g - 2}$. We consider

$$f \circ_i g(\mathbf{x}, \mathbf{y}, \mathbf{z}) = f(\mathbf{x}, g(\mathbf{y}), \mathbf{z}). \quad (\star)$$

If (\star) is zero, then it is in \mathcal{A}_1 . Namely (deg2-1) is satisfied. So we assume $(\star) \neq 0$. We consider the case of $g(\mathbf{y}) \in \mathcal{A}_1$. In this case, \mathbf{y} is in $\mathcal{A}^{l_g, k_g - 1}$ and $\mathbf{x} \otimes \mathbf{z}$ is in $\mathcal{A}^{l_f - 1, k_f - 1}$. Thus $\mathbf{x} \otimes g(\mathbf{y}) \otimes \mathbf{z}$ is an element in $\mathcal{A}^{l_f, k_f - 1}$ which implies $f(\mathbf{x} \otimes g(\mathbf{y}) \otimes \mathbf{z}) \in \mathcal{A}_1$. When the case of $g(\mathbf{y}) \in \mathcal{A}_2$, \mathbf{y} is in $\mathcal{A}^{l_g - 1, k_g}$ and $\mathbf{x} \otimes \mathbf{z}$ is in $\mathcal{A}^{l_f, k_f - 2}$. Thus $\mathbf{x} \otimes g(\mathbf{y}) \otimes \mathbf{z}$ is an element in $\mathcal{A}^{l_f, k_f - 1}$ which gives $f(\mathbf{x} \otimes g(\mathbf{y}) \otimes \mathbf{z}) \in \mathcal{A}_1$. Similar way, when $\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}$ is an element in $\mathcal{A}^{l_f + l_g - 2, k_f + k_g - 1}$, the condition holds. We show (deg3). If $\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}$ is an element in $\mathcal{A}^{l_f + l_g - 1 + i, k_f + k_g - 2 - i}$ and $g(\mathbf{y}) \neq 0$, then $\mathbf{x} \otimes g(\mathbf{y}) \otimes \mathbf{z}$ is in $\mathcal{A}^{l_f + i, k_f - 1 - i}$. When $i \neq 0$, from the assumption, $f(\mathbf{x} \otimes g(\mathbf{y}) \otimes \mathbf{z}) = 0$. The proof is completed. \square

Proposition 2.6. *If $\|f\| = k_f l_f$ and $\|g\| = k_g l_g$, then the Gerstenhaber bracket $\{f, g\}$ has the bidegree $k_f + k_g - 1 | l_f + l_g - 1$.*

Proof. Straightforward. \square

Remark. Given a bidegree $k + 1 | l + 1$ -cochain f , we define $\text{bideg}(f) := k | l$. If $\text{bideg}(f) = k | l$ and $\text{bideg}(g) = m | n$, then $\text{bideg}(\{f, g\}) = \text{bideg}(f) + \text{bideg}(g) = k + m | l + n$. Thus the bidegree, bideg , of Gerstenhaber bracket is $0 | 0$.

3 Main objects.

Notations. Let \mathcal{A}_1 and \mathcal{A}_2 be vector spaces. We denote any elements of \mathcal{A}_1 by a, b, c, \dots and denote any elements of \mathcal{A}_2 by x, y, z, \dots . We sometimes use an identification $(a, x) \cong a + x$, where $(a, x) \in \mathcal{A}_1 \oplus \mathcal{A}_2$.

3.1 Twilled algebras.

3.1.1 Structures.

Let \mathcal{T} be an associative algebra equipped with an associative structure θ . We assume a decomposition of \mathcal{T} into two subspaces, $\mathcal{T} = \mathcal{A}_1 \oplus \mathcal{A}_2$. The associative structure defines an associative multiplication by $\theta((a, x), (b, y)) := (a, x) * (b, y)$, for any $(a, x), (b, y) \in \mathcal{T}$.

Definition 3.1. ([5]) *The triple $(\mathcal{T}, \mathcal{A}_1, \mathcal{A}_2)$, or simply \mathcal{T} , is called an associative **twilled algebra**, if \mathcal{A}_1 and \mathcal{A}_2 are subalgebras of \mathcal{T} . We sometimes denote a twilled algebra \mathcal{T} by $\mathcal{A}_1 \bowtie \mathcal{A}_2$.*

One can easily check that if $\mathcal{A}_1 \bowtie \mathcal{A}_2$ is a twilled algebra, then \mathcal{A}_1 (resp. \mathcal{A}_2) is an \mathcal{A}_2 -bimodule (resp. \mathcal{A}_1 -bimodule). These bimodule structures are defined by the following decomposition of associative multiplication of \mathcal{T} . For any $a \in \mathcal{A}_1$ and $x \in \mathcal{A}_2$, the multiplications $a * x$ and $x * a$ are decomposed into 4 multiplications,

$$a * x = (a *_2 x, a *_1 x), \quad x * a = (x *_2 a, x *_1 a),$$

where $a *_2 x$ and $x *_2 a$ are \mathcal{A}_1 -components of $a * x$ and $x * a$ respectively, and similar way, $a *_1 x$ and $x *_1 a$ are \mathcal{A}_2 -components. One can easily check that the multiplication $*_1$ (resp. $*_2$) is the bimodule action of \mathcal{A}_1 to \mathcal{A}_2 (resp. \mathcal{A}_2 to \mathcal{A}_1).

In general, the associative multiplication of $\mathcal{A}_1 \bowtie \mathcal{A}_2$ has the form,

$$(a, x) * (b, y) = (a * b + a *_2 y + x *_2 b, a *_1 y + x *_1 b + x * y).$$

The total multiplication, $*$, is decomposed into two “associative” multiplications of semidirect product,

$$\begin{aligned} (a, x) *_1 (b, y) &:= (a *_1 b, a *_1 y + x *_1 b), \\ (a, x) *_2 (b, y) &:= (a *_2 y + x *_2 b, x *_2 y), \end{aligned}$$

where we put $a *_1 b := a * b$ and $x *_2 y := x * y$. Hence the structure θ is also decomposed into two associative structures,

$$\theta = \hat{\mu}_1 + \hat{\mu}_2,$$

where $\hat{\mu}_i$ is the structure associated with the multiplication $*_i$ for $i = 1, 2$. Recall (8). The cochains $\hat{\mu}_1$ and $\hat{\mu}_2$ have the bidegrees $1|2$ and $2|1$ respectively. Under the assumption, the decomposition of θ is unique, i.e., if θ is decomposed into two substructures of bidegrees $1|2$ and $2|1$, then such substructures are uniquely determined.

Lemma 3.2. *The associativity of θ ($\{\theta, \theta\} = 0$) is equivalent with the compatibility conditions,*

$$\frac{1}{2} \{\hat{\mu}_1, \hat{\mu}_1\} = 0, \quad (9)$$

$$\{\hat{\mu}_1, \hat{\mu}_2\} = 0, \quad (10)$$

$$\frac{1}{2} \{\hat{\mu}_2, \hat{\mu}_2\} = 0. \quad (11)$$

Proof. We will show a more generalized result in Lemma 3.9 below. \square

3.1.2 The case of subalgebras in duality

Given an arbitrary associative algebra \mathcal{A} , we have a Lie algebra by the commutator, $[a, b] := ab - ba$ on \mathcal{A} . The induced Lie algebra is denoted by $L(\mathcal{A})$. The correspondence $L : \mathcal{A} \rightarrow L(\mathcal{A})$ is a functor (sometimes called a Liezation) from the usual category of associative algebras to the one of Lie algebras.

In this short section, we assume that $\mathcal{A}_1 =: \mathcal{A}$ is a finite dimensional vector space and \mathcal{A}_2 is the dual space. In this case, $\mathcal{T} = \mathcal{A} \oplus \mathcal{A}^*$ has a nondegenerate symmetric bilinear form, $(-|-)$, where $(\mathcal{A}|\mathcal{A}^*) = (\mathcal{A}^*|\mathcal{A})$ is the dual pairing and $(\mathcal{A}|\mathcal{A}) = (\mathcal{A}^*|\mathcal{A}^*) = 0$. We set a natural assumption, namely, the bilinear form is invariant (or associative) with respect to the associative multiplication of \mathcal{T} , explicitly,

$$(t_1 * t_2 | t_3) = (t_1 | t_2 * t_3)$$

for any $t_1, t_2, t_3 \in \mathcal{T}$. Such a twilled algebra is called an invariant twilled algebra.

If \mathcal{T} is an invariant twilled algebra, then the triple $(L(\mathcal{T}), L(\mathcal{A}), L(\mathcal{A}^*))$ is a Manin triple. It is a twilled Lie algebra with an invariant pseudo-Euclidean metric decomposed into two maximally isotropic subalgebras. In general, a pair of Lie algebras $(\mathfrak{g}_1, \mathfrak{g}_2)$ becomes a Lie bialgebra if and only if a triple of Lie algebras $(\mathfrak{g}_1 \bowtie \mathfrak{g}_2, \mathfrak{g}_1, \mathfrak{g}_2)$ is a Manin triple. In this times, the total space $\mathfrak{g}_1 \bowtie \mathfrak{g}_2$ is called a Drinfeld double. Thus the pair $(L(\mathcal{A}), L(\mathcal{A}^*))$ becomes a Lie bialgebra and $L(\mathcal{A}) \bowtie L(\mathcal{A}^*)$ is a Drinfeld double. If \mathcal{T} is a quasi-twilled algebra in Definition 3.10 below, then the cocycle term ϕ_1 (or ϕ_2) is a cyclic cocycle, i.e., for any $a, b, c \in \mathcal{A}$,

$$\phi_1(a, b)(c) = \phi_1(b, c)(a) = \phi_1(c, a)(b).$$

This fact is directly checked by the invariancy. And the commutator, $\Phi_1(a, b) := \phi_1(a, b) - \phi_1(b, a)$, is identified with a skew symmetric 3-tensor in $\bigwedge^3 \mathcal{A}^*$. This implies that if $\mathcal{A} \oplus \mathcal{A}^*$ is a quasi-twilled algebra, then $L(\mathcal{T})$ is the double of quasi-Lie bialgebra $(L(\mathcal{A}), L(\mathcal{A}^*))$ (see [7], [12] for quasi-Lie bialgebras).

The dual map of an associative multiplication on \mathcal{T} becomes a coassociative multiplication $\mathcal{T} \rightarrow \mathcal{T} \otimes \mathcal{T}$. Here \mathcal{T} and $\mathcal{T} \otimes \mathcal{T}$ are identified with \mathcal{T}^* and $(\mathcal{T} \otimes \mathcal{T})^*$ by the bilinear form. Since $\hat{\mu}_i$ is associative, the dual map of $\hat{\mu}_i$ becomes a coassociative multiplication, $\Delta_{\hat{\mu}_i} : \mathcal{T} \rightarrow \mathcal{T} \otimes \mathcal{T}$, $i = 1, 2$. We rewrite the conditions (9), (10) and (11) by the comultiplications. (9) and (11) are equivalent with coassociativity of $\Delta_{\hat{\mu}_i}$, $i = 1, 2$, respectively. So we consider (10). We define a $(\mathcal{T}, \hat{\mu}_1)$ -bimodule structure on $\mathcal{T} \otimes \mathcal{T}$ by $t \cdot (\mathcal{T} \otimes \mathcal{T}) := (t *_{\mathcal{T}} \mathcal{T}) \otimes \mathcal{T}$ and $(\mathcal{T} \otimes \mathcal{T}) \cdot t := \mathcal{T} \otimes (\mathcal{T} *_{\mathcal{T}} t)$ where $t \in \mathcal{T}$ and $*_{\mathcal{T}}$ is the associative multiplication of $\hat{\mu}_1$. For any $s, t, u, v \in \mathcal{T}$, we have

$$(\Delta_{\hat{\mu}_2}(s *_{\mathcal{T}} t)|u \otimes v) = (s *_{\mathcal{T}} t|u *_{\mathcal{T}} v),$$

where the pairing $(-|-)$ is extended on $\mathcal{T} \otimes \mathcal{T}$ by the rule,

$$(s \otimes t|u \otimes v) := (s|v)(t|u).$$

The invariancy holds with respect to $\hat{\mu}_i$, $i = 1, 2$, for instance,

$$(a *_{\mathcal{T}} x|b) = (a * x|b) = (a|x * b) = (a|x *_{\mathcal{T}} b),$$

where $(\mathcal{A}|\mathcal{A}) = 0$ is used. From the invariancy, we have $(s *_{\mathcal{T}} t|u *_{\mathcal{T}} v) = (s|t *_{\mathcal{T}} (u *_{\mathcal{T}} v))$. By (10), we have $t *_{\mathcal{T}} (u *_{\mathcal{T}} v) = (t *_{\mathcal{T}} u) *_{\mathcal{T}} v + (t *_{\mathcal{T}} u) *_{\mathcal{T}} v - t *_{\mathcal{T}} (u *_{\mathcal{T}} v)$. Thus (10) is equivalent with the condition,

$$\begin{aligned} (\Delta_{\hat{\mu}_2}(s *_{\mathcal{T}} t)|u \otimes v) &= (s|t *_{\mathcal{T}} (u *_{\mathcal{T}} v)) = \\ &= (s|(t *_{\mathcal{T}} u) *_{\mathcal{T}} v) + (s|(t *_{\mathcal{T}} u) *_{\mathcal{T}} v) - (s|t *_{\mathcal{T}} (u *_{\mathcal{T}} v)). \end{aligned} \quad (12)$$

The first term of the right-hand side of (12) is

$$(s|(t *_{\mathcal{T}} u) *_{\mathcal{T}} v) = (v *_{\mathcal{T}} s|t *_{\mathcal{T}} u) = (u *_{\mathcal{T}} (v *_{\mathcal{T}} s)|t) = (u \otimes (v *_{\mathcal{T}} s)|\Delta_{\hat{\mu}_2}(t)).$$

We put $\Delta_{\hat{\mu}_2}(t) = \sum t_1 \otimes t_2$. Then we have

$$(u \otimes (v *_{\mathcal{T}} s)|\Delta_{\hat{\mu}_2}(t)) = \sum (u|t_2)(v *_{\mathcal{T}} s|t_1) = \sum (u|t_2)(v|s *_{\mathcal{T}} t_1) = (u \otimes v|s \cdot \Delta_{\hat{\mu}_2}(t)). \quad (A)$$

And the second and third terms of the right-hand side of (12) are

$$(s|(t *_1 u) *_2 v) - (s|t *_2 (u *_1 v)) = (\Delta_{\hat{\mu}_2}(s)|(t *_1 u) \otimes v) - (s *_2 t|u *_1 v).$$

We put $\Delta_{\hat{\mu}_2}(s) = \sum s_1 \otimes s_2$. Then we have

$$(\Delta_{\hat{\mu}_2}(s)|(t *_1 u) \otimes v) = \sum (s_1|v)(s_2|t *_1 u) = \sum (s_1|v)(s_2 *_1 t|u) = (\Delta_{\hat{\mu}_2}(s) \cdot t|u \otimes v). \quad (B)$$

and

$$(s *_2 t|u *_1 v) = (\Delta_{\hat{\mu}_1}(s *_2 t)|u \otimes v) = (\Delta_{\hat{\mu}_1} \circ \hat{\mu}_2(s, t)|u \otimes v). \quad (C)$$

From (A), (B) and (C), we obtain a compatibility condition,

$$(\Delta_{\hat{\mu}_2}(s *_1 t)|u \otimes v) = (s \cdot \Delta_{\hat{\mu}_2}(t)|u \otimes v) + (\Delta_{\hat{\mu}_2}(s) \cdot t|u \otimes v) - (\Delta_{\hat{\mu}_1} \circ \hat{\mu}_2(s, t)|u \otimes v). \quad (13)$$

Since $\mathcal{T} \otimes \mathcal{T}$ is a $(\mathcal{T}, \hat{\mu}_1)$ -bimodule, we have a Hochschild complex $(C^*(\mathcal{T}, \mathcal{T} \otimes \mathcal{T}), D_{\hat{\mu}_1})$, where $D_{\hat{\mu}_1}$ is a Hochschild coboundary map. The condition (13) is equivalent with (14) below. Under the assumptions of this section, the identity (10) $\{\hat{\mu}_1, \hat{\mu}_2\} = 0$ is equivalent with

$$D_{\hat{\mu}_1} \Delta_{\hat{\mu}_2} - \Delta_{\hat{\mu}_1} \circ \hat{\mu}_2 = 0. \quad (14)$$

Since $\{\hat{\mu}_2, \hat{\mu}_1\} = 0$, we have $D_{\hat{\mu}_2} \Delta_{\hat{\mu}_1} - \Delta_{\hat{\mu}_2} \circ \hat{\mu}_1 = 0$. One can easily show that $D_{\hat{\mu}_i} \Delta_{\hat{\mu}_i} - \Delta_{\hat{\mu}_i} \circ \hat{\mu}_i = 0$ holds for $i = 1, 2$. Thus we have $D_\theta \Delta_\theta - \Delta_\theta \circ \theta = 0$. From (14) we have $D_{\hat{\mu}_1}(\Delta_1 \circ \hat{\mu}_2) = 0$. By direct computation, one can show that if \mathcal{A} is unital (i.e. $1 *_1 \mathcal{A} = \mathcal{A} *_1 1$), then $D_{\hat{\mu}_1}(\Delta_1 \circ \hat{\mu}_2) = 0$ implies (14).

It is obvious that \mathcal{A} is a sub-coalgebra of $(\mathcal{T}, \Delta_{\hat{\mu}_2})$. Since $\hat{\mu}_2$ is zero on $\mathcal{A} \otimes \mathcal{A}$, $\Delta_{\hat{\mu}_2}$ is a derivation on \mathcal{A} , i.e., for any $a, b \in \mathcal{A}$,

$$\Delta_{\hat{\mu}_2}(a *_1 b) = \Delta_{\hat{\mu}_2}(a) \cdot b + a \cdot \Delta_{\hat{\mu}_2}(b).$$

An associative and coassociative algebra $(\mathcal{I}, *, \delta)$ is called an infinitesimal bialgebra ([9]), if $\delta(a * b) = a \cdot \delta(b) + \delta(b) \cdot a$ for any $a, b \in \mathcal{I}$. Thus the triple $(\mathcal{A}, *_1, \Delta_{\hat{\mu}_2})$ is an infinitesimal bialgebra. We consider the converse. Given an infinitesimal bialgebra $(\mathcal{I}, *, \delta)$, the multiplications $*$ and δ are extended on $\mathcal{I} \oplus \mathcal{I}^*$ by adjoint actions. However the compatibility condition (14) is not satisfied in general. This implies that the Liezation of an infinitesimal bialgebra is not a Lie bialgebra in general. For this problem, see the detailed study Aguiar [3].

3.1.3 Induced dg-Lie algebras.

This short section is the heart of this article. The meaning of twilled algebra is given by the proposition below. From the associative condition (9), $(C^*(\mathcal{T}), d_{\hat{\mu}_1}(-) := \{\hat{\mu}_1, -\})$ becomes a dg-Lie algebra. The graded space $C^*(\mathcal{A}_2, \mathcal{A}_1)$ is identified with an abelian subalgebra of the dg-Lie algebra, via the horizontal lift. One can easily check that the derived bracket

$$[f, g]_{\hat{\mu}_1} := (-1)^{|f|-1} \{\{\hat{\mu}_1, f\}, g\}$$

is closed on $C^*(\mathcal{A}_2, \mathcal{A}_1)$. From Lemma 2.1, $C^*(\mathcal{A}_2, \mathcal{A}_1)$ becomes a graded Lie algebra. Further, by (10) and (11), $d_{\hat{\mu}_2} := \{\hat{\mu}_2, \}$ becomes a square zero derivation on the induced graded Lie algebra $C^*(\mathcal{A}_2, \mathcal{A}_1)$.

Proposition 3.3. *If $\mathcal{T} = \mathcal{A}_1 \bowtie \mathcal{A}_2$ is a twilled algebra, then $C^*(\mathcal{A}_2, \mathcal{A}_1)$ has a dg-Lie algebra structure. The degree of dg-Lie algebra structure is the same as the usual degree of cochains.*

Proof. We show only a derivation property of $d_{\hat{\mu}_2}$. Since $\hat{\mu}_2$ is an associative structure, $d_{\hat{\mu}_2}$ is square zero. For any cochains $f, g \in C^*(\mathcal{A}_2, \mathcal{A}_1)$, we have

$$\begin{aligned} d_{\hat{\mu}_2}[f, g]_{\hat{\mu}_1} &:= (-1)^{|f|-1} \{\hat{\mu}_2, \{\{\hat{\mu}_1, f\}, g\}\} \\ &= (-1)^{|f|-1} \{\{\hat{\mu}_2, \{\hat{\mu}_1, f\}\}, g\} - \{\{\hat{\mu}_1, f\}, \{\hat{\mu}_2, g\}\} \\ &= (-1)^{|f|} \{\{\hat{\mu}_1, \{\hat{\mu}_2, f\}\}, g\} - \{\{\hat{\mu}_1, f\}, \{\hat{\mu}_2, g\}\} \\ &= [d_{\hat{\mu}_2}f, g]_{\hat{\mu}_1} + (-1)^{|f|}[f, d_{\hat{\mu}_2}g]_{\hat{\mu}_1}. \end{aligned}$$

From Lemma 2.1, the derived degree is given by $\deg_{d_{\hat{\mu}_1}}(f) = \deg(f) + 1 = |f|$, where $\deg(f) = |f| - 1$ is the degree of the canonical dg-Lie algebra $(C^*(\mathcal{T}), d_{\hat{\mu}_1})$ (recall Section 2.1). Thus $d_{\hat{\mu}_2}$ satisfies the defining condition (3) of dg-Lie algebra. \square

When we recall deformation theory, it is natural to ask: What is a solution of Maurer-Cartan equation in the dg-Lie algebra? We will solve this question in Section 5.

3.1.4 Examples

Example 3.4. (*trivial extensions, semidirect product algebras.*) Let \mathcal{A} be an associative algebra and let M an \mathcal{A} -bimodule. The trivial extension $\mathcal{A} \ltimes M$ is a twilled algebra of $\mathcal{A} = \mathcal{A}_1$ and $M = \mathcal{A}_2$, where the structure $\hat{\mu}_2$ is trivial and $\hat{\mu}_1$ is defined by, for any $(a, m), (b, n) \in \mathcal{A} \oplus M$,

$$\hat{\mu}_1((a, m), (b, n)) := (a, m) * (b, n) := (ab, a \cdot n + m \cdot b),$$

where \cdot is the bimodule action of \mathcal{A} on M .

A direct product algebra $\mathcal{A} \times \mathcal{A}$ is a twilled algebra. The following example is considered as a q -analogue of trivial extensions.

Example 3.5. (*q -trivial extensions.*) Let \mathcal{A} be an associative algebra. Define a multiplication on $\mathcal{A} \oplus \mathcal{A}$ by

$$(a, x) *_q (b, y) := (ab, ay + xb + qxy),$$

where $q \in \mathbb{K}$. Then $(\mathcal{A} \oplus \mathcal{A}, *_q)$ becomes a twilled algebra. We denote the twilled algebra by $\mathcal{A} \bowtie_q \mathcal{A}$.

If (\mathcal{T}, θ) is an associative algebra, then $C^*(\mathcal{T})$ becomes an associative algebra by a cup product, $f \vee_\theta g := \theta(f, g)$, $f, g \in C^*(\mathcal{T})$.

Example 3.6. *If $\mathcal{T} = \mathcal{A}_1 \bowtie \mathcal{A}_2$ is a twilled algebra, then*

$$C^*(\mathcal{T}) = C^*(\mathcal{T}, \mathcal{A}_1 \bowtie \mathcal{A}_2) \cong C^*(\mathcal{T}, \mathcal{A}_1) \bowtie C^*(\mathcal{T}, \mathcal{A}_2)$$

is a twilled algebra, because the cup product is decomposed into $\vee_\theta = \vee_{\hat{\mu}_1} + \vee_{\hat{\mu}_2}$.

3.2 Proto-, Quasi-twilled algebras.

A *quasi-Lie* bialgebra is known as a classical limit of a quasi-Hopf algebra. The notion of quasi-Lie bialgebra is generalized to *proto-Lie* bialgebras (see [12]). The latter is more complicated object than quasi-Lie bialgebras. The proto-Lie bialgebras provide a general framework of quantum-classical correspondence. In this section, we will study associative analogues of proto-, quasi-Lie bialgebras.

Definition 3.7. Let (\mathcal{T}, θ) be an associative algebra decomposed into two subspaces, $\mathcal{T} = \mathcal{A}_1 \oplus \mathcal{A}_2$. Here \mathcal{A}_1 and \mathcal{A}_2 are not necessarily subalgebras. We call the triple $(\mathcal{T}, \mathcal{A}_1, \mathcal{A}_2)$ a **proto-twilled algebra**.

Lemma 3.8. Let θ be an arbitrary 2-cochain in $C^2(\mathcal{T})$. Then θ is uniquely decomposed into 4 homogeneous cochains of bidegrees $0|3$, $1|2$, $2|1$ and $3|0$,

$$\theta = \hat{\phi}_1 + \hat{\mu}_1 + \hat{\mu}_2 + \hat{\phi}_2.$$

Proof. Recall the decomposition (7). The space of 2-cochains $C^2(\mathcal{T})$ is decomposed into 4 subspaces,

$$C^2(\mathcal{T}) = (0|3) \oplus (1|2) \oplus (2|1) \oplus (3|0),$$

where $(i|j)$ is the space of bidegree $i|j$ -cochains, $i, j = 0, 1, 2, 3$. The decomposition is essentially unique. Thus θ is uniquely decomposed into homogeneous cochains of bidegrees $0|3$, $1|2$, $2|1$ and $3|0$. The 4 substructures $\hat{\phi}_1$, $\hat{\mu}_1$, $\hat{\mu}_2$ and $\hat{\phi}_2$ in the lemma are given as the homogeneous cochains. The proof is completed. \square

The multiplication $(a, x) * (b, y) := \theta((a, x), (b, y))$ of \mathcal{T} is uniquely decomposed by the canonical projections $\mathcal{T} \rightarrow \mathcal{A}_1$ and $\mathcal{T} \rightarrow \mathcal{A}_2$ into the 8 multiplications,

$$\begin{aligned} a * b &= (a *_1 b, a *_2 b), \\ a * y &= (a *_2 y, a *_1 y), \\ x * b &= (x *_2 b, x *_1 b), \\ x * y &= (x *_1 y, x *_2 y). \end{aligned}$$

We put bidegrees on the 4 cochains, $\|\hat{\phi}_1\| := 0|3$, $\|\hat{\mu}_1\| := 1|2$, $\|\hat{\mu}_2\| := 2|1$ and $\|\hat{\phi}_2\| := 3|0$. Then we obtain

$$\begin{aligned} \hat{\phi}_1((a, x), (b, y)) &= (0, a *_2 b), \\ \hat{\mu}_1((a, x), (b, y)) &= (a *_1 b, a *_1 y + x *_1 b), \\ \hat{\mu}_2((a, x), (b, y)) &= (a *_2 y + x *_2 b, x *_2 y), \\ \hat{\phi}_2((a, x), (b, y)) &= (x *_1 y, 0). \end{aligned}$$

Remark that $\hat{\phi}_1$ and $\hat{\phi}_2$ are lifted cochains of $\phi_1(a, b) := a *_2 b$ and $\phi_2(x, y) := x *_1 y$.

Lemma 3.9. *The Maurer-Cartan condition $\{\theta, \theta\} = 0$ is equivalent with the following 5 conditions.*

$$\{\hat{\mu}_1, \hat{\phi}_1\} = 0, \quad (15)$$

$$\frac{1}{2}\{\hat{\mu}_1, \hat{\mu}_1\} + \{\hat{\mu}_2, \hat{\phi}_1\} = 0, \quad (16)$$

$$\{\hat{\mu}_1, \hat{\mu}_2\} + \{\hat{\phi}_1, \hat{\phi}_2\} = 0, \quad (17)$$

$$\frac{1}{2}\{\hat{\mu}_2, \hat{\mu}_2\} + \{\hat{\mu}_1, \hat{\phi}_2\} = 0, \quad (18)$$

$$\{\hat{\mu}_2, \hat{\phi}_2\} = 0. \quad (19)$$

Proof. From the 5 conditions, one can directly check the Maurer-Cartan condition of θ . We show the converse. The bidegrees of $\hat{\phi}_1$, $\hat{\mu}_1$, $\hat{\mu}_2$ and $\hat{\phi}_2$ are $0|3$, $1|2$, $2|1$ and $3|0$, respectively. If $\{\theta, \theta\} = 0$, then

$$\begin{aligned} \{\hat{\mu}_1, \hat{\mu}_1\} + 2\{\hat{\mu}_2, \hat{\phi}_1\} + 2\{\hat{\mu}_1, \hat{\mu}_2\} + 2\{\hat{\phi}_1, \hat{\phi}_2\} + \{\hat{\mu}_2, \hat{\mu}_2\} + 2\{\hat{\mu}_1, \hat{\phi}_2\} + \\ 2\{\hat{\mu}_1, \hat{\phi}_1\} + 2\{\hat{\mu}_2, \hat{\phi}_2\} = 0. \end{aligned}$$

The first two terms have $1|3$ -bidegree, the second two terms have $2|2$ -bidegree, the third two terms have $3|1$ -bidegree and the last two terms have $0|4$ and $4|0$ respectively. Thus we have $\{\hat{\mu}_1, \hat{\mu}_1\} + 2\{\hat{\mu}_2, \hat{\phi}_1\} = 0$ for $1|3$ -bidegree, and this is (16). Similarly, we obtain (15)-(19). \square

Definition 3.10. *Let $\mathcal{T} = \mathcal{A}_1 \oplus \mathcal{A}_2$ be a proto-twilled algebra equipped with the structures $(\hat{\mu}_1, \hat{\mu}_2, \hat{\phi}_1, \hat{\phi}_2)$. We call the triple $(\mathcal{T}, \mathcal{A}_1, \mathcal{A}_2)$ a **quasi-twilled algebra**, if $\phi_2 = 0$, or equivalently, \mathcal{A}_2 is a subalgebra. Since $\mathcal{A}_1 \oplus \mathcal{A}_2 = \mathcal{A}_2 \oplus \mathcal{A}_1$, the definition is adapted in the case of $\phi_2 \neq 0$ and $\phi_1 = 0$.*

It is obvious that twilled algebras are special quasi-twilled algebras of $\phi_1 = \phi_2 = 0$. From Lemma 3.9, θ is the structure of a quasi-twilled algebra of $\phi_2 = 0$ if and only if

$$\{\hat{\mu}_1, \hat{\phi}_1\} = 0, \quad (20)$$

$$\frac{1}{2}\{\hat{\mu}_1, \hat{\mu}_1\} + \{\hat{\mu}_2, \hat{\phi}_1\} = 0, \quad (21)$$

$$\{\hat{\mu}_1, \hat{\mu}_2\} = 0, \quad (22)$$

$$\frac{1}{2}\{\hat{\mu}_2, \hat{\mu}_2\} = 0. \quad (23)$$

In Proposition 3.3, we saw $C^*(\mathcal{A}_2, \mathcal{A}_1)$ has a dg-Lie algebra structure. In the quasi-twilled algebra cases, from (23), $d_{\hat{\mu}_2}$ is still a square zero derivation, but the derived bracket by $\hat{\mu}_1$ does not satisfy the graded Jacobi identity in general. However the Jacobiator still satisfies a weak Jacobi identity in the sense of homotopy Lie algebras ([6],[15]). The 3-cochain $\frac{1}{2}\{\hat{\mu}_1, \hat{\mu}_1\}$ rises up to the graded Jacobiator via the derived bracket,

$$\begin{aligned} (-1)^{|g|-1} \frac{1}{2} \{ \{ \{ \hat{\mu}_1, \hat{\mu}_1 \}, f \}, g \}, h \} = \\ [f, [g, h]_{\hat{\mu}_1}]_{\hat{\mu}_1} - [[f, g]_{\hat{\mu}_1}, h]_{\hat{\mu}_1} - (-1)^{|f||g|} [g, [f, h]_{\hat{\mu}_1}]_{\hat{\mu}_1}. \end{aligned}$$

From (21), the Jacobiator is also given by $-\{\hat{\mu}_2, \hat{\phi}_1\}$. We define a tri-linear bracket product (homotopy) on $C^*(\mathcal{A}_2, \mathcal{A}_1)$ by

$$[f, g, h]_{\hat{\phi}_1} := (-1)^{|g|-1} \{\{\{\hat{\phi}_1, f\}, g\}, h\}.$$

Since $C^*(\mathcal{A}_2, \mathcal{A}_1)$ is abelian with respect to $\{-, -\}$, the tribracket is skew-symmetric. We can show that the system, $(d_{\hat{\mu}_2}, [\cdot, \cdot]_{\hat{\mu}_1}, [\cdot, \cdot, \cdot]_{\hat{\phi}_1})$, defines a strong homotopy Lie algebra structure of $l_{n \geq 4} := 0$ on $C^*(\mathcal{A}_2, \mathcal{A}_1)$. This assertion will be shown as a corollary of a more general result in [24].

The complex plane, $\mathcal{T} := \mathbb{C}$, is a quasi-twilled algebra decomposed into the real part and the imaginary part. Given a \mathbb{R} -algebra \mathcal{A} , the complexification $\mathbb{C} \otimes_{\mathbb{R}} \mathcal{A} = \mathcal{A} \oplus \sqrt{-1}\mathcal{A}$ is a quasi-twilled algebra.

Example 3.11. (*Quasi-trivial extension.*) Let \mathcal{A} be an associative algebra. Define a multiplication on $\mathcal{A} \oplus \mathcal{A}$ by

$$(a, x) *_Q (b, y) := (ab + Qxy, ay + xb),$$

where $Q \in \mathbb{K}$. Then $\mathcal{A} \oplus \mathcal{A}$ becomes a quasi-twilled algebra, where $\phi_2(x, y) := Qxy$. We denote the algebra by $\mathcal{A} \oplus_Q \mathcal{A}$.

4 Twisting by a 1-cochain

Let h be a 1-cochain in $C^1(\mathcal{T})$. By analogy with Hamiltonian vector field, we define an operator by $X_h := \{\cdot, h\}$, and by analogy with Hamiltonian flow, we put

$$\exp(X_h)(\cdot) := 1 + X_h + \frac{1}{2!}X_h^2 + \frac{1}{3!}X_h^3 + \dots,$$

where $X_h^2 := \{\{\cdot, h\}, h\}$ and X_h^n is defined by the same manner. Remark that $\exp(X_h)$ is not well-defined in general.

Let $(\mathcal{T} = \mathcal{A}_1 \oplus \mathcal{A}_2, \theta)$ be a proto-twilled algebra, and let $\hat{H} \in C^1(\mathcal{T})$ be the lift of a linear map $H : \mathcal{A}_2 \rightarrow \mathcal{A}_1$ (or $H : \mathcal{A}_1 \rightarrow \mathcal{A}_2$). Then $\exp(X_{\hat{H}})$ is always well-defined as an operator, because $\hat{H}\hat{H} = 0$ (recall Lemma 2.2).

Definition 4.1. A transformation (24) is called a “twisting” of θ by H .

$$\theta^H := \exp(X_{\hat{H}})(\theta). \quad (24)$$

It is clear that the result of twisting by H is again a 2-cochain. We can consider the twisting operations are special examples of gauge transformations in deformation theory (see [6]). The following Lemma 4.2 and Proposition 4.3 are followed from standard arguments in deformation theory.

Lemma 4.2. $\theta^H = e^{-\hat{H}}\theta(e^{\hat{H}} \otimes e^{\hat{H}})$, where $e^{\pm \hat{H}} = 1 \pm \hat{H}$.

Proof. We have $e^{-\hat{H}}\theta(e^{\hat{H}} \otimes e^{\hat{H}}) = \theta(e^{\hat{H}} \otimes e^{\hat{H}}) - \hat{H}\theta(e^{\hat{H}} \otimes e^{\hat{H}}) =$

$$\begin{aligned} &= \theta + \theta(1 \otimes \hat{H}) + \theta(\hat{H} \otimes 1) + \theta(\hat{H} \otimes \hat{H}) - \hat{H}\theta - \hat{H}\theta(1 \otimes \hat{H}) - \hat{H}\theta(\hat{H} \otimes 1) - \hat{H}\theta(\hat{H} \otimes \hat{H}) = \\ &\theta + \theta(1 \otimes \hat{H}) + \theta(\hat{H} \otimes 1) - \hat{H}\theta + \theta(\hat{H} \otimes \hat{H}) - \hat{H}\theta(1 \otimes \hat{H}) - \hat{H}\theta(\hat{H} \otimes 1) - \hat{H}\theta(\hat{H} \otimes \hat{H}). \end{aligned}$$

Since $\hat{H}\hat{H} = 0$, for any $I \geq 4$, we have $X_{\hat{H}}^I(\theta) = 0$. Thus we have

$$\exp(X_{\hat{H}})(\theta) = \theta + \{\theta, \hat{H}\} + \frac{1}{2}\{\{\theta, \hat{H}\}, \hat{H}\} + \frac{1}{6}\{\{\{\theta, \hat{H}\}, \hat{H}\}, \hat{H}\}.$$

One can directly check the three identities below.

$$\begin{aligned} \{\theta, \hat{H}\} &= \theta(\hat{H} \otimes 1) + \theta(1 \otimes \hat{H}) - \hat{H}\theta, \\ \frac{1}{2}\{\{\theta, \hat{H}\}, \hat{H}\} &= \theta(\hat{H} \otimes \hat{H}) - \hat{H}\theta(\hat{H} \otimes 1) - \hat{H}\theta(1 \otimes \hat{H}), \\ \frac{1}{6}\{\{\{\theta, \hat{H}\}, \hat{H}\}, \hat{H}\} &= -\hat{H}\theta(\hat{H} \otimes \hat{H}). \end{aligned}$$

The proof of the lemma is completed. \square

From above lemma, we have $\{\theta^H, \theta^H\} = e^{-H}\{\theta, \theta\}(e^{\hat{H}} \otimes e^{\hat{H}} \otimes e^{\hat{H}})$. This implies

Proposition 4.3. *The result of twisting θ^H is an associative structure, i.e., $\{\theta^H, \theta^H\} = 0$.*

The following corollary is useful.

Corollary 4.4. *The twisting by H induces an algebra isomorphism,*

$$e^H : (\mathcal{T}, \theta^H) \rightarrow (\mathcal{T}, \theta).$$

Obviously, (\mathcal{T}, θ^H) is also a proto-twilled algebra. Thus θ^H is also decomposed into the unique 4 substructures. The twisting operations are completely determined by

Theorem 4.5. *Assume a decomposition of θ , $\theta := \hat{\mu}_1 + \hat{\mu}_2 + \hat{\phi}_1 + \hat{\phi}_2$. The unique 4 substructures of θ^H have the following form:*

$$\hat{\phi}_1^H = \hat{\phi}_1, \quad (25)$$

$$\hat{\mu}_1^H = \hat{\mu}_1 + \{\hat{\phi}_1, \hat{H}\}, \quad (26)$$

$$\hat{\mu}_2^H = \hat{\mu}_2 + d_{\hat{\mu}_1}\hat{H} + \frac{1}{2}\{\{\hat{\phi}_1, \hat{H}\}, \hat{H}\}, \quad (27)$$

$$\hat{\phi}_2^H = \hat{\phi}_2 + d_{\hat{\mu}_2}\hat{H} + \frac{1}{2}[\hat{H}, \hat{H}]_{\hat{\mu}_1} + \frac{1}{6}\{\{\{\hat{\phi}_1, \hat{H}\}, \hat{H}\}, \hat{H}\}, \quad (28)$$

where $d_{\hat{\mu}_i}(-) := \{\hat{\mu}_i, -\}$, $(i = 1, 2)$ and $[\hat{H}, \hat{H}]_{\hat{\mu}_1} := \{\{\hat{\mu}_1, \hat{H}\}, \hat{H}\}$.

Proof. The first term of $\exp(X_{\hat{H}})(\theta)$ is θ . From the bidegree calculus, we have $\{\hat{\phi}_2, \hat{H}\} = 0$, because $\|\hat{\phi}_2\| = 3|0$ and $\|\hat{H}\| = 2|0$. Thus the second term of $\exp(X_{\hat{H}})(\theta)$ has the form,

$$\{\hat{\mu}_1, \hat{H}\} + \{\hat{\mu}_2, \hat{H}\} + \{\hat{\phi}_1, \hat{H}\}.$$

We have $\|\{\hat{\mu}_1, \hat{H}\}\| = 2|1$, $\|\{\hat{\mu}_2, \hat{H}\}\| = 3|0$ and $\|\{\hat{\phi}_1, \hat{H}\}\| = 1|2$, which implies $\{\{\hat{\mu}_2, \hat{H}\}, \hat{H}\} = 0$. Thus the third term has the form,

$$\frac{1}{2}(\{\{\hat{\mu}_1, \hat{H}\}, \hat{H}\} + \{\{\hat{\phi}_1, \hat{H}\}, \hat{H}\}).$$

The bidegrees are $\|\{\hat{\mu}_1, \hat{H}\}, \hat{H}\| = 3|0$ and $\|\{\{\hat{\phi}_1, \hat{H}\}, \hat{H}\}\| = 2|1$. The final term is $\{\{\{\theta, \hat{H}\}, \hat{H}\}, \hat{H}\} = \{\{\{\hat{\phi}_1, \hat{H}\}, \hat{H}\}, \hat{H}\}$ which has the bidegree $3|0$. Thus the sum of all $3|0$ -terms is

$$\hat{\phi}_2 + \{\hat{\mu}_2, \hat{H}\} + \frac{1}{2!}\{\{\hat{\mu}_1, \hat{H}\}, \hat{H}\} + \frac{1}{3!}\{\{\{\hat{\phi}_1, \hat{H}\}, \hat{H}\}, \hat{H}\}$$

which gives (28). In this way, the remaining 3 conditions hold. \square

5 Maurer-Cartan equations

Let $\mathcal{T} = \mathcal{A}_1 \oplus \mathcal{A}_2$ be a proto-twilled algebra equipped with an associative structure θ and let $(\hat{\phi}_1, \hat{\mu}_1, \hat{\mu}_2, \hat{\phi}_2)$ be the unique 4 substructures of θ . In this section, we discuss various examples of twisting operations.

5.1 The cases of $\phi_1 = 0$ and $\phi_2 = 0$.

In this case, $\mathcal{T} = \mathcal{A}_1 \bowtie \mathcal{A}_2$ is a twilled algebra. However the result of twisting by $H : \mathcal{A}_2 \rightarrow \mathcal{A}_1$, $(\mathcal{T}_H, \mathcal{A}_1, \mathcal{A}_2)$, is a quasi-twilled algebra in general. The twisted structures have the forms,

$$\begin{aligned}\hat{\mu}_1^H &= \hat{\mu}_1, \\ \hat{\mu}_2^H &= \hat{\mu}_2 + d_{\hat{\mu}_1} \hat{H}, \\ \hat{\phi}_2^H &= d_{\hat{\mu}_2} \hat{H} + \frac{1}{2}[\hat{H}, \hat{H}]_{\hat{\mu}_1}.\end{aligned}$$

This $\hat{\phi}_2^H$ is called a curvature. The derivation operator $d_{\hat{\mu}_2}$ on the graded Lie algebra $C^*(\mathcal{A}_2, \mathcal{A}_1)$ is modified by H , $d_{\hat{\mu}_2^H}(-) = d_{\hat{\mu}_2}(-) + [\hat{H}, -]_{\hat{\mu}_1}$, where $d_{\hat{\mu}_2^H} d_{\hat{\mu}_2^H} \neq 0$ in general. By Lemma 3.9 (19), the cocycle condition of ϕ_2^H still holds,

$$d_{\hat{\mu}_2^H} \hat{\phi}_2^H = 0.$$

This is a kind of Bianchi identity.

5.1.1 Maurer-Cartan operators.

In Proposition 3.3, we saw that $C^*(\mathcal{A}_2, \mathcal{A}_1)$ has a dg-Lie algebra structure. We study a Maurer-Cartan equation in the dg-Lie algebra.

Corollary 5.1. *The result of twisting $\mathcal{T}_H = \mathcal{A}_1 \oplus \mathcal{A}_2$ is also a twilled algebra if and only if the curvature vanishes, or equivalently, H is a solution of a Maurer-Cartan equation,*

$$d_{\hat{\mu}_2} \hat{H} + \frac{1}{2}[\hat{H}, \hat{H}]_{\hat{\mu}_1} = 0. \quad (MC)$$

The condition (MC) is equivalent with

$$H(x) *_1 H(y) + H(x) *_2 y + x *_2 H(y) = H(H(x) *_1 y + x *_1 H(y)) + H(x *_2 y). \quad (29)$$

Proof. We have $d_{\hat{\mu}_2} \hat{H} = \hat{\mu}_2(\hat{H} \otimes 1) - \hat{H} \hat{\mu}_2 + \hat{\mu}_2(1 \otimes \hat{H})$ and

$$\begin{aligned} \frac{1}{2}[\hat{H}, \hat{H}]_{\hat{\mu}_1} &= \frac{1}{2}\{\{\hat{\mu}_1, \hat{H}\}, \hat{H}\} \\ &= \hat{\mu}_1(\hat{H} \otimes \hat{H}) - \hat{H} \hat{\mu}_1(1 \otimes \hat{H}) - \hat{H} \hat{\mu}_1(\hat{H} \otimes 1). \end{aligned}$$

This gives, for any $(a, x), (b, y) \in \mathcal{T}$,

$$\begin{aligned} (d_{\hat{\mu}_2} \hat{H} + \frac{1}{2}[\hat{H}, \hat{H}]_{\hat{\mu}_1})(a, x), (b, y)) = \\ H(x) *_2 y - H(x *_2 y) + x *_2 H(y) + H(x) *_1 H(y) - H(H(x) *_1 y + x *_1 H(y)). \end{aligned}$$

□

Definition 5.2. Let $\mathcal{A}_1 \bowtie \mathcal{A}_2$ be a twilled algebra and let $H : \mathcal{A}_2 \rightarrow \mathcal{A}_1$ a linear map. We call the operator H in (MC), or equivalently, in (29) a **Maurer-Cartan operator**. A Maurer-Cartan operator is called **strong**, if it is a derivation with respect to the multiplication $*_2$, i.e.,

$$H(x *_2 y) = x *_2 H(y) + H(x) *_2 y.$$

In Liu and coauthors [16], a Maurer-Cartan equation in other dg-Lie algebra was studied. The concept of strong solution is due to their work. If H is strong, then the identity, $H(x) *_1 H(y) = H(H(x) *_1 y + x *_1 H(y))$, automatically holds. The strong Maurer-Cartan condition is equivalent with

$$d_{\hat{\mu}_2} \hat{H} = \frac{1}{2}[\hat{H}, \hat{H}]_{\hat{\mu}_1} = 0.$$

We easily obtain

Corollary 5.3. If H is a Maurer-Cartan operator, then

$$x \times_H y := H(x) *_1 y + x *_1 H(y) + x *_2 y$$

is an associative multiplication on \mathcal{A}_2 .

Proof. When H satisfies (MC), we have $\hat{\phi}_2^H = 0$. By Lemma 3.9, we obtain $\{\hat{\mu}_2^H, \hat{\mu}_2^H\} = 0$ which gives the associativity of $\hat{\mu}_2^H$. The multiplication has the following form on \mathcal{A}_2 ,

$$\hat{\mu}_2^H(x, y) = H(x) *_1 y + x *_1 H(y) + x *_2 y.$$

□

We recall Rota-Baxter operators in Introduction.

Example 5.4. (Rota-Baxter operators of weight q .) Let \mathcal{A} be an associative algebra. We recall the twilled algebra in Example 3.5. The multiplication of $\mathcal{A} \bowtie_q \mathcal{A}$ is defined by

$$(a, x) *_q (b, y) := (ab, ay + xb + qxy), \quad (30)$$

where $q \in \mathbb{K}$ (weight). From (29), the Maurer-Cartan operators on $\mathcal{A} \bowtie_q \mathcal{A}$ satisfy the Rota-Baxter identity of weight q ,

$$R(x)R(y) = R(R(x)y + xR(y)) + qR(xy).$$

where we put $R := H$. Thus Rota-Baxter operators can be seen as examples of Maurer-Cartan operators.

As an example of Rota-Baxter operator, we know

$$R(f)(x) := f(qx) + f(q^2x) + f(q^3x) + \dots \quad (\text{convergent})$$

where R is defined on a certain algebra of functions (see [19]).

5.1.2 The cases of $\hat{\mu}_2 = 0$.

Consider the cases of $\hat{\mu}_2 = 0$. In this case, since $d_{\hat{\mu}_2} = 0$, the Maurer-Cartan equation simply has the form, $[\hat{H}, \hat{H}]_{\hat{\mu}_1}/2 = 0$, or equivalently, (29) reduces to the identity,

$$H(x) *_1 H(y) = H(H(x) *_1 y + x *_1 H(y)).$$

Further, if $\mathcal{A}_2 = \mathcal{A}_1$ as a canonical bimodule, then H is considered as a Rota-Baxter operator with weight zero.

Definition 5.5. ([23]) Let \mathcal{A} be an associative algebra and let M be an \mathcal{A} -bimodule. A linear map $\pi : M \rightarrow \mathcal{A}$ is called a generalized Rota-Baxter operator (of weight zero), if π is a solution of the identity,

$$\pi(m)\pi(n) = \pi(\pi(m) \cdot n + m \cdot \pi(n)), \quad (31)$$

or equivalently, $[\hat{\pi}, \hat{\pi}]_{\hat{\mu}}/2 = 0$, where $m, n \in M$ and $\hat{\mu}$ is the associative structure of $\mathcal{A} \ltimes M$.

A generalized Rota-Baxter operator is obviously a (strong-)Maurer-Cartan operator. Given a generalized Rota-Baxter operator $\pi : M \rightarrow \mathcal{A}$, we have a twilled algebra $\mathcal{A} \ltimes M_\pi$ by the twisting of $\mathcal{A} \ltimes M$ by π , where M_π is an associative subalgebra given by Corollary 5.3. The associative structure of $\mathcal{A} \ltimes M_\pi$ is the sum of two structures, $\hat{\mu} + \{\hat{\mu}, \hat{\pi}\}$.

Corollary 5.6. Under the assumptions above, if π_1 is a second generalized Rota-Baxter operator on $\mathcal{A} \ltimes M$, i.e., $[\hat{\pi}_1, \hat{\pi}_1]_{\hat{\mu}} = 0$, then $H := \pi_1 - \pi$ is a Maurer-Cartan operator on $\mathcal{A} \ltimes M_\pi$. If H is strong, then $\pi + tH$ is a one parameter family of generalized Rota-Baxter operators for any $t \in \mathbb{K}$.

Proof. From assumptions, we have $[\hat{H}, \hat{H}]_{\hat{\mu}}/2 = -[\hat{\pi}_1, \hat{\pi}]_{\hat{\mu}}$. On the other hand, since $d_{\hat{\mu}_2}(\cdot) = \{\{\hat{\mu}, \hat{\pi}\}, \cdot\}$, we have

$$d_{\hat{\mu}_2}\hat{H} = \{\{\hat{\mu}, \hat{\pi}\}, \hat{\pi}_1\} = [\hat{\pi}, \hat{\pi}_1]_{\hat{\mu}} = [\hat{\pi}_1, \hat{\pi}]_{\hat{\mu}}.$$

Simply, we obtain the condition (MC). Thus Maurer-Cartan operators on $\mathcal{A} \ltimes M_\pi$ are given as the difference of π with generalized Rota-Baxter operators. If H is a strong Maurer-Cartan operator, then tH is also so for any $t \in \mathbb{K}$. This implies the second part of the corollary. \square

We recall in Section 3.1.2. Let \mathcal{A} be a finite dimensional associative algebra and let \mathcal{A}^* the dual space. By a canonical adjoint action, \mathcal{A} acts on the dual space. In this case, there are interesting similarities in between generalized Rota-Baxter operators and classical r -matrices. We recall classical Yang-Baxter equation (CYBE). There exists several equivalent definition

of CYBE. We recall the one of them. CYBE is defined to be an operator identity in the category of Lie algebras,

$$[\tilde{r}(x), \tilde{r}(y)] = \tilde{r}([\tilde{r}(x), y] + [x, \tilde{r}(y)])$$

where r is a two tensor in $\mathfrak{g} \otimes \mathfrak{g}$ (\mathfrak{g} is a finite dimensional Lie algebra), $\tilde{r} : \mathfrak{g}^* \rightarrow \mathfrak{g}$ is the associated linear map, x, y are elements in the dual space \mathfrak{g}^* and the brackets in the right-hand side are adjoint actions. The space of alternative tensors $\bigwedge^2 \mathfrak{g}$ has a graded Lie algebra structure of Schouten bracket. If r is an element in $\mathfrak{g} \wedge \mathfrak{g}$, then the Schouten bracket $[r, r]$ is in $\bigwedge^3 \mathfrak{g}$, and $[r, r] = 0$ if and only if \tilde{r} satisfies CYBE above. Such a matrix r is called a triangular r -matrix. When \mathfrak{g} is a Lie algebroid, a triangular r -matrix is a Poisson structure. The notion of generalized Rota-Baxter operator can be seen as an associative version of triangular r -matrices and Poisson structures. We believe that this picture is justified by the following example.

Example 5.7. Let \mathcal{A} be a 2-dimensional algebra generated by $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. The dual space \mathcal{A}^* is an \mathcal{A} -bimodule by adjoint action. Thus we have a twilled algebra $\mathcal{A} \ltimes \mathcal{A}^*$. Define a tensor r by

$$r := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The tensor r is identified with a map $\tilde{r} : \mathcal{A}^* \rightarrow \mathcal{A}$. By direct computation, one can check that the map is a generalized Rota-Baxter operator.

In general, if a 2-tensor $r \in \mathcal{A} \wedge \mathcal{A}$ satisfies Aguiar's multiplicative equation (called an associative Yang-Baxter) in [1, 2, 3],

$$r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13} = 0, \quad (AYBE)$$

then $\tilde{r} : \mathcal{A}^* \rightarrow \mathcal{A}$ is a generalized Rota-Baxter operator (see [23]). Conversely, a skew symmetric generalized Rota-Baxter operator satisfies (AYBE) above. In non skewsymmetric cases, there is a delicate difference between AYBE and the generalized Rota-Baxter condition.

When r is skewsymmetric, the twisting by r preserves the bilinear pairing $(-|-)$ in Section 3.1.2. Thus the associative structure $\hat{\mu} + \{\hat{\mu}, \hat{r}\}$ satisfies the invariant condition in the sense of 3.1.2.

A Poisson structure is considered as a sheaf version of triangular matrices. It is natural to ask what is a sheaf version of Rota-Baxter operators. We do not yet have an interesting solution. We wish to find a Rota-Baxter operator on the universal enveloping algebra of a Lie algebroid. If there exists such a Rota-Baxter operator, it is considered as an example of the sheaf version.

5.2 The cases of $\phi_1 \neq 0$ and $\phi_2 = 0$.

In this case, $\mathcal{T} = \mathcal{A}_1 \oplus \mathcal{A}_2$ is a quasi-twilled algebra. However $\mathcal{T}_H = \mathcal{A}_1 \oplus \mathcal{A}_2$ is not necessarily a quasi-twilled algebra, because $\phi_1^H = \phi_1 \neq 0$ and

$$\hat{\phi}_2^H = d_{\hat{\mu}_2} \hat{H} + \frac{1}{2} [\hat{H}, \hat{H}]_{\hat{\mu}_1} + \frac{1}{6} \{ \{ \{ \hat{\phi}_1, \hat{H} \}, \hat{H} \}, \hat{H} \} \neq 0.$$

In general, the result of twisting have the forms,

$$\begin{aligned}\hat{\phi}_1^H &= \hat{\phi}_1, \\ \hat{\mu}_1^H &= \hat{\mu}_1 + \{\hat{\phi}_1, \hat{H}\}, \\ \hat{\mu}_2^H &= \hat{\mu}_2 + d_{\hat{\mu}_1} \hat{H} + \frac{1}{2} \{\{\hat{\phi}_1, \hat{H}\}, \hat{H}\}, \\ \hat{\phi}_2^H &= d_{\hat{\mu}_2} \hat{H} + \frac{1}{2} [\hat{H}, \hat{H}]_{\hat{\mu}_1} + \frac{1}{6} \{\{\{\hat{\phi}_1, \hat{H}\}, \hat{H}\}, \hat{H}\},\end{aligned}$$

Since $\hat{\mu}_1$ is not associative, the derived bracket $[\cdot, \cdot]_{\hat{\mu}_1}$ does not satisfy the graded Jacobi rule in general. However the space $C^*(\mathcal{A}_2, \mathcal{A}_1)$ still has a homotopy Lie algebra structure $(d_{\hat{\mu}_2}, [\cdot, \cdot]_{\hat{\mu}_1}, [\cdot, \cdot, \cdot]_{\hat{\phi}_1})$ in Section 3.2. We consider a Maurer-Cartan equation in this homotopy Lie algebra. The following two corollaries are followed by the same manners with Corollary 5.1 and Corollary 5.3.

Corollary 5.8. *The result of twisting $\mathcal{T}_H = \mathcal{A}_1 \oplus \mathcal{A}_2$ is also a quasi-twilled algebra if and only if it is a solution of twisted Maurer-Cartan equation,*

$$d_{\hat{\mu}_2} \hat{H} + \frac{1}{2} [\hat{H}, \hat{H}]_{\hat{\mu}_1} + \frac{1}{6} [\hat{H}, \hat{H}, \hat{H}]_{\hat{\phi}_1} = 0, \quad (TMC)$$

or equivalently, for any $x, y \in \mathcal{A}_2$,

$$\begin{aligned}H(x) *_1 H(y) + H(x) *_2 y + x *_2 H(y) = \\ H(H(x) *_1 y + x *_1 H(y)) + H(x *_2 y) + H(\phi_1(H(x), H(y))).\end{aligned} \quad (32)$$

Corollary 5.9. *If $\mathcal{T}_H = \mathcal{A}_1 \oplus \mathcal{A}_2$ is a quasi-twilled algebra, then*

$$x \times_{H, \phi_1} y := \hat{\mu}_2^H(x, y) = H(x) *_1 y + x *_1 H(y) + x *_2 y + \phi_1(H(x), H(y)).$$

is an associative multiplication on \mathcal{A}_2 .

Example 5.10. *(Twisted Rota-Baxter operators [23].) If $\hat{\mu}_2 = 0$, or $*_2$ is trivial, then (32) is reduced to an identity:*

$$H(x) *_1 H(y) = H(H(x) *_1 y + x *_1 H(y)) + H(\phi_1(H(x), H(y))). \quad (TRB1)$$

(TRB1) is equivalent with

$$\frac{1}{2} [\hat{H}, \hat{H}]_{\hat{\mu}_1} = -\frac{1}{6} [\hat{H}, \hat{H}, \hat{H}]_{\hat{\phi}_1}. \quad (TRB2)$$

Such an operator H is called a twisted Rota-Baxter operator (of weight zero).

As an example of twisted Rota-Baxter operators, we know Reynolds operators in probability theory ([20]). Let \mathcal{A} be a certain functional algebra. Define an operator $R : \mathcal{A} \rightarrow \mathcal{A}$ by

$$R(f)(x) := \int_0^\infty e^{-t} f(x-t) dt.$$

Then R satisfies an identity,

$$R(f)R(g) = R(R(f)g + fR(g)) - R(R(f)R(g)).$$

Such an operator is called a Reynolds operator. The last term $-R(R(f)R(g)) = R\phi(R(f), R(g))$ can be seen as the cocycle term of twisted Rota-Baxter identity. Thus a Reynolds operator can be seen as a homotopy version of Rota-Baxter operators of weight zero.

A Reynolds operator is used, in the study of turbulent flow, in order to induce a mean field model of Navier-Stokes equation (so-called Reynolds equation). One can easily verify that if $R(f) := \overline{f}$ is the mean of f , then the operator satisfies the identity above, because an averaging operation satisfies the identities $\overline{\overline{f}g} = \overline{f} \cdot \overline{g} = \overline{f\overline{g}}$ and $\overline{\overline{f}} = \overline{f}$ in general. Unfortunately, we do not know an application of our construction to Rota's theory.

5.3 The cases of $\phi_1 = 0$ and $\phi_2 \neq 0$

In this case, $\hat{\phi}_1 = \hat{\phi}_1^H = 0$, and thus $\hat{\mu}_1$ and $\hat{\mu}_1^H$ are both associative. The twisted 4 substructures have the forms,

$$\begin{aligned}\hat{\mu}_1^H &= \hat{\mu}_1, \\ \hat{\mu}_2^H &= \hat{\mu}_2 + d_{\hat{\mu}_1} \hat{H}, \\ \hat{\phi}_2^H &= \hat{\phi}_2 + d_{\hat{\mu}_2} \hat{H} + \frac{1}{2}[\hat{H}, \hat{H}]_{\hat{\mu}_1}.\end{aligned}$$

Similar with Corollary 5.1 and Corollary 5.3, we obtain the two corollaries below.

Corollary 5.11. *The result of twisting $\mathcal{T}_H = \mathcal{A}_1 \oplus \mathcal{A}_2$ is a usual twilled algebra, i.e., $\hat{\phi}_2^H = 0$ if and only if H is a solution of the quasi-Maurer-Cartan equation,*

$$d_{\hat{\mu}_2} \hat{H} + \frac{1}{2}[\hat{H}, \hat{H}]_{\hat{\mu}_1} = -\hat{\phi}_2, \quad (QMC)$$

or equivalently,

$$\begin{aligned}H(x) *_2 y + x *_2 H(y) + H(x) *_1 H(y) + \phi_2(x, y) = \\ H(H(x) *_1 y + x *_1 H(y)) + H(x *_2 y).\end{aligned} \quad (33)$$

Corollary 5.12. *If H satisfied (QMC), then $\hat{\mu}_2^H$ is an associative structure and defines an associative multiplication on \mathcal{A}_2 by*

$$x \times_{H, \phi_2} y := \hat{\mu}_2^H(x, y) = H(x) *_1 y + x *_1 H(y) + x *_2 y. \quad (34)$$

We consider a case of $\hat{\mu}_2 = 0$. Then (QMC) and (33) reduce to the identities, respectively,

$$\frac{1}{2}[\hat{H}, \hat{H}]_{\hat{\mu}_1} = -\hat{\phi}_2,$$

and

$$H(x) *_1 H(y) - H(H(x) *_1 y + x *_1 H(y)) = -\phi_2(x, y). \quad (35)$$

Recall the quasi-twilled algebra $\mathcal{A} \oplus_Q \mathcal{A}$ in Example 3.11.

Claim. Define a linear map $(a, x) \mapsto (\frac{q}{2}x, 0)$ on $\mathcal{A} \oplus \mathcal{A}$. Then its integral $\widehat{e^{q/2}}$ is an algebra isomorphism,

$$\widehat{e^{q/2}} : \mathcal{A} \bowtie_q \mathcal{A} \rightarrow \mathcal{A} \oplus_Q \mathcal{A}, \quad Q = \frac{q^2}{4}.$$

Proof.

$$\begin{aligned}
e^{\widehat{q/2}}((a, x) *_q (b, y)) &= (ab + \frac{q}{2}ay + \frac{q}{2}xb + \frac{q^2}{2}xy, ay + xb + qxy) \\
&= ((a + \frac{q}{2}x)(b + \frac{q}{2}y) + \frac{q^2}{4}xy, ay + xb + qxy) \\
&= (a + \frac{q}{2}x, x) *_Q (b + \frac{q}{2}y, y), \quad Q = \frac{q^2}{4}.
\end{aligned}$$

□

If $Q = 0$, then $\mathcal{A} \oplus_{Q=0} \mathcal{A}$ is the semi-direct product algebra. Thus $\mathcal{A} \bowtie_q \mathcal{A}$ is isomorphic with $\mathcal{A} \ltimes \mathcal{A}$ modulus q^2 .

Now, the claim says that $\mathcal{A} \bowtie_q \mathcal{A}$ is the result of twisting of $\mathcal{A} \oplus_Q \mathcal{A}$ by $q/2$. Let $(R(\mathcal{A}), \mathcal{A})$ be the graph of R . One can easily verify that if R is a q -Rota-Baxter operator, then $\mathcal{A} \bowtie_q \mathcal{A} = \mathcal{A} \bowtie (R(\mathcal{A}), \mathcal{A})$ is a second twilled algebra decomposition. By the twisting, we have a twilled algebra, $\mathcal{A} \bowtie (R(\mathcal{A}) + \frac{q}{2}\mathcal{A}, \mathcal{A})$,

$$\mathcal{A} \bowtie (R(\mathcal{A}), \mathcal{A}) = \mathcal{A} \bowtie_q \mathcal{A} \xrightarrow{e^{\widehat{q/2}}} \mathcal{A} \oplus_{q^2/4} \mathcal{A} = \mathcal{A} \bowtie (R(\mathcal{A}) + \frac{q}{2}\mathcal{A}, \mathcal{A}).$$

Example 5.13. (*Rota-Baxter operator mod q^2 [8]*). Let (\mathcal{A}, R) be a Rota-Baxter algebra. We define a linear map $B : \mathcal{A} \rightarrow \mathcal{A}$ by $B(\mathcal{A}) := R(\mathcal{A}) + \frac{q}{2}\mathcal{A}$. Then the graph of B , $(B(\mathcal{A}), \mathcal{A})$, is a subalgebra of the quasi-twilled algebra $\mathcal{A} \oplus_{q^2/4} \mathcal{A}$. This implies that B is a solution of

$$B(x)B(y) - B(B(x)y + xB(y)) = -\frac{q^2}{4}xy.$$

The right-hand term $q^2/4xy := \phi_2(x, y)$ can be seen as the cocycle-term in (35).

6 Application.

In this section, we will give a construction of associative Nijenhuis operator. First we recall basic properties of Nijenhuis operator. A linear operator, $N : \mathcal{A} \rightarrow \mathcal{A}$, is called an associative Nijenhuis operator, if N is a solution of

$$N(x)N(y) = N(N(x)y + xN(y)) - N^2(xy).$$

In general, given a Nijenhuis operator, $x \times_N y := N(x)y + xN(y) - N(xy)$ is a second associative multiplication and it is compatible with the original multiplication. Namely, $xy + tx \times_N y$ is a one parameter family of associative multiplications for any $t \in \mathbb{K}$ ([5]).

In the following, we assume that \mathcal{A} is an associative algebra, M is an \mathcal{A} -bimodule and we denote the multiplication of \mathcal{A} by $*_{\mathcal{A}}$.

Let $\pi : M \rightarrow \mathcal{A}$ be a generalized Rota-Baxter operator, i.e., π satisfies the identity,

$$\pi(m) *_{\mathcal{A}} \pi(n) = \pi(\pi(m) \cdot n + m \cdot \pi(n)). \quad (36)$$

where \cdot is the bimodule action of \mathcal{A} on M and $m, n \in M$. We recall the twilled algebra $\mathcal{A} \bowtie M_\pi$ in Section 5.1.3. The associative multiplication of $\mathcal{A} \bowtie M_\pi$ has the form

$$(a, m) * (b, n) = (a *_{\mathcal{A}} b + a \cdot_{\pi} n + m \cdot_{\pi} b, a \cdot n + m \cdot b + m \times_{\pi} n),$$

where \cdot_{π} means the bimodule action of M_π on \mathcal{A} , explicitly,

$$\begin{aligned} m \cdot_{\pi} b &:= \pi(m) *_{\mathcal{A}} b - \pi(m \cdot b), \\ a \cdot_{\pi} n &:= a *_{\mathcal{A}} \pi(n) - \pi(a \cdot n), \end{aligned}$$

and $m \times_{\pi} n$ is the associative multiplication of M_π , explicitly,

$$m \times_{\pi} n := \pi(m) \cdot n + m \cdot \pi(n).$$

Simply, we have $\pi(m \times_{\pi} n) = \pi(m) *_{\mathcal{A}} \pi(n)$.

We consider a linear map $\Omega : \mathcal{A} \rightarrow M_\pi$. The map Ω is a strong Maurer-Cartan operator on a twilled algebra $M_\pi \bowtie \mathcal{A}$ if and only if

$$\Omega(a *_{\mathcal{A}} b) = a \cdot \Omega(b) + \Omega(a) \cdot b, \quad (37)$$

$$\Omega(a) \times_{\pi} \Omega(b) = \Omega(\Omega(a) \cdot_{\pi} b + a \cdot_{\pi} \Omega(b)), \quad (38)$$

or equivalently, Ω is a solution of

$$d_{\hat{\mu}} \hat{\Omega} = \frac{1}{2} [\hat{\Omega}, \hat{\Omega}]_{\{\hat{\mu}, \hat{\pi}\}} = 0.$$

We give the main result of this section.

Proposition 6.1. *Let $\Omega : \mathcal{A} \rightarrow M_\pi$ be a strong Maurer-Cartan operator.*

1. *Then a composition map $N := \pi\Omega$ is an associative Nijenhuis operator on \mathcal{A} . Namely N satisfies the condition*

$$N(a) *_{\mathcal{A}} N(b) = N(N(a) *_{\mathcal{A}} b + a *_{\mathcal{A}} N(b)) - NN(a *_{\mathcal{A}} b)$$

for any $a, b \in \mathcal{A}$.

The pair of (π, N) is compatible in the following sense.

2. *The composition $N\pi : M \rightarrow \mathcal{A}$ is a second generalized Rota-Baxter operator.*
3. *The operators π and $N\pi$ are compatible, i.e.,*

$$[\hat{\pi}, \widehat{N\pi}]_{\hat{\mu}} = 0.$$

This implies that $N\pi$ is strong as a Maurer-Cartan operator and $\pi + tN\pi$ $t \in \mathbb{K}$ is a one parameter family of generalized Rota-Baxter operators.

Proof. 1. Applying π to (38), we have

$$\pi\Omega(a) *_{\mathcal{A}} \pi\Omega(b) = \pi\Omega(\Omega(a) \cdot_{\pi} b + a \cdot_{\pi} \Omega(b)).$$

In the right-hand side,

$$\Omega(a) \cdot_{\pi} b + a \cdot_{\pi} \Omega(b) = \pi\Omega(a) *_{\mathcal{A}} b - \pi(\Omega(a) \cdot b) + a *_{\mathcal{A}} \pi\Omega(b) - \pi(a \cdot \Omega(b)).$$

From (37), we have

$$\Omega(a) \cdot_{\pi} b + a \cdot_{\pi} \Omega(b) = \pi\Omega(a) *_{\mathcal{A}} b + a *_{\mathcal{A}} \pi\Omega(b) - \pi\Omega(a *_{\mathcal{A}} b).$$

Thus we obtain the desired condition,

$$\pi\Omega(a) *_{\mathcal{A}} \pi\Omega(b) = \pi\Omega(\pi\Omega(a) *_{\mathcal{A}} b + a *_{\mathcal{A}} \pi\Omega(b)) - \pi\Omega\pi\Omega(a *_{\mathcal{A}} b).$$

2. We put $a := \pi(m)$ and $b := \pi(n)$ for any $m, n \in M$. Then, by the Nijenhuis condition of $\pi\Omega$, we have

$$\pi\Omega\pi(m) *_{\mathcal{A}} \pi\Omega\pi(n) = \pi\Omega(\pi\Omega\pi(m) *_{\mathcal{A}} \pi(n) + \pi(m) *_{\mathcal{A}} \pi\Omega\pi(n)) - \pi\Omega\pi\Omega(\pi(m) *_{\mathcal{A}} \pi(n)). \quad (39)$$

From the identity (36), we have

$$\begin{aligned} \pi\Omega\pi(m) *_{\mathcal{A}} \pi(n) &= \pi(\pi\Omega\pi(m) \cdot n + \Omega\pi(m) \cdot \pi(n)), \\ \pi(m) *_{\mathcal{A}} \pi\Omega\pi(n) &= \pi(\pi(m) \cdot \Omega\pi(n) + m \cdot \pi\Omega\pi(n)), \end{aligned}$$

and from the derivation rule, we have

$$\pi\Omega\pi\Omega(\pi(m) *_{\mathcal{A}} \pi(n)) = \pi\Omega\pi(\Omega\pi(m) \cdot \pi(n) + \pi(m) \cdot \Omega\pi(n)).$$

Thus (39) has the form,

$$\begin{aligned} \pi\Omega\pi(m) *_{\mathcal{A}} \pi\Omega\pi(n) &= \pi\Omega\pi(\pi\Omega\pi(m) \cdot n + \Omega\pi(m) \cdot \pi(n) + \pi(m) \cdot \Omega\pi(n) + m \cdot \pi\Omega\pi(n)) - \\ &\quad \pi\Omega\pi(\Omega\pi(m) \cdot \pi(n) + \pi(m) \cdot \Omega\pi(n)) = \\ &\quad \pi\Omega\pi(\pi\Omega\pi(m) \cdot n + m \cdot \pi\Omega\pi(n)), \end{aligned}$$

this is the desired result.

3. It is obvious that $\widehat{\pi\Omega\pi} = \hat{\pi}\hat{\Omega}\hat{\pi}$. We have $[\hat{\pi}, \widehat{\pi\Omega\pi}]_{\hat{\mu}} =$

$$\begin{aligned} \{\hat{\mu}, \hat{\pi}\}, \hat{\pi}\hat{\Omega}\hat{\pi} &= \{\hat{\mu}(\pi \otimes 1) + \hat{\mu}(1 \otimes \hat{\pi}) - \hat{\pi}\hat{\mu}, \hat{\pi}\hat{\Omega}\hat{\pi}\} = \\ &\quad \hat{\mu}(\hat{\pi} \otimes \hat{\pi}\hat{\Omega}\hat{\pi}) - \hat{\pi}\hat{\Omega}\hat{\pi}\hat{\mu}(\hat{\pi} \otimes 1) + \hat{\mu}(\hat{\pi}\hat{\Omega}\pi \otimes \hat{\pi}) - \hat{\pi}\hat{\Omega}\hat{\pi}\hat{\mu}(1 \otimes \hat{\pi}) \\ &\quad - \hat{\pi}\hat{\mu}(\hat{\pi}\hat{\Omega}\hat{\pi} \otimes 1) - \hat{\pi}\hat{\mu}(1 \otimes \hat{\pi}\hat{\Omega}\hat{\pi}), \quad (40) \end{aligned}$$

where $\hat{\pi}\hat{\pi} = 0$ is used. From the generalized Rota-Baxter condition, $[\hat{\pi}, \hat{\pi}]_{\hat{\mu}}/2 = \hat{\mu}(\hat{\pi} \otimes \hat{\pi}) - \hat{\pi}\hat{\mu}(\hat{\pi} \otimes 1) - \hat{\pi}\hat{\mu}(1 \otimes \hat{\pi}) = 0$, we have

$$\begin{aligned} (40) &= \hat{\mu}(\hat{\pi} \otimes \hat{\pi}\hat{\Omega}\hat{\pi}) - \hat{\pi}\hat{\Omega}\hat{\mu}(\hat{\pi} \otimes \hat{\pi}) + \hat{\mu}(\hat{\pi}\hat{\Omega}\pi \otimes \hat{\pi}) - \hat{\pi}\hat{\mu}(\hat{\pi}\hat{\Omega}\hat{\pi} \otimes 1) - \hat{\pi}\hat{\mu}(1 \otimes \hat{\pi}\hat{\Omega}\hat{\pi}) = \\ &\quad - \hat{\pi}\hat{\Omega}\hat{\mu}(\hat{\pi} \otimes \hat{\pi}) + \hat{\mu}(\hat{\pi}\hat{\Omega}\pi \otimes \hat{\pi}) - \hat{\pi}\hat{\mu}(\hat{\pi}\hat{\Omega}\hat{\pi} \otimes 1) + \hat{\pi}\hat{\mu}(\hat{\pi} \otimes \hat{\Omega}\hat{\pi}) = \\ &\quad - \hat{\pi}\hat{\Omega}\hat{\mu}(\hat{\pi} \otimes \hat{\pi}) + \hat{\pi}\hat{\mu}(\hat{\Omega}\hat{\pi} \otimes \hat{\pi}) + \hat{\pi}\hat{\mu}(\hat{\pi} \otimes \hat{\Omega}\hat{\pi}). \quad (41) \end{aligned}$$

Since $\hat{\Omega}$ is a derivation with respect to $\hat{\mu}$, the last equation of (41) is zero. \square

Example 6.2. We put $\mathcal{A} := C^1([0, 1])$ and $M := C^0([0, 1])$. We assume a canonical bimodule action of \mathcal{A} on M . An integral operator is a Rota-Baxter operator with weight zero.

$$\pi : M \rightarrow \mathcal{A}, \quad \pi(f)(x) := \int_0^x dt f(t).$$

Then a derivation from \mathcal{A} to M_π ,

$$\Omega(f)(x) := \omega(x) \frac{df}{dx}(x) = \omega(x) f'(x), \quad \omega(x) \in C^0([0, 1])$$

is a strong Maurer-Cartan operator. The induced Nijenhuis operator on \mathcal{A} is

$$N(f)(x) = \int_0^x \omega(t) f'(t) dt.$$

Proof. We only check the condition (38). For any $f, g \in \mathcal{A}$,

$$\Omega(f) \cdot_\pi g = \pi \Omega(f)g - \pi(\Omega(f)g) = \int_0^x dt \omega(t) f'(t) g(x) - \int_0^x dt \omega(t) f'(t) g(t).$$

We have

$$\begin{aligned} \Omega(\Omega(f) \cdot_\pi g) &= \int_0^x dt \omega(t) f'(t) \omega(x) g'(x), \\ \Omega(f \cdot_\pi \Omega(g)) &= \omega(x) f'(x) \int_0^x dt \omega(t) g'(t). \end{aligned}$$

On the other hand,

$$\begin{aligned} \Omega(f) \times_\pi \Omega(g) &= \omega(x) f'(x) \times_\pi \omega(x) g'(x) \\ &= \int_0^x dt \omega(t) f'(t) \omega(x) g'(x) + \omega(x) f'(x) \int_0^x dt \omega(t) g'(t). \end{aligned}$$

Thus we obtain the desired condition. \square

We consider two examples in noncommutative cases. In the proof of Example 6.2, we used the commutativity of only ω . Hence if ω is 1 or a central element, then the similar proof holds over noncommutative setting.

Example 6.3. Let \mathcal{A} be an associative algebra and let $\mathcal{A}[[\nu]]$ an algebra of formal series with coefficients in \mathcal{A} . The multiplication on $\mathcal{A}[[\nu]]$ is defined by

$$a_i \nu^i * b_j \nu^j := a_i b_j \nu^{i+j}, \quad a_i, b_j \in \mathcal{A},$$

where \sum is omitted. We define a formal integral operator,

$$\int d\nu a_i \nu^i := \frac{1}{i+1} a_i \nu^{i+1}, \quad a_i \in \mathcal{A}.$$

The integral operator is a Rota-Baxter operator with weight zero. The formal derivation operator is a strong Maurer-Cartan operator

$$\Omega(a_i \nu^i) := z_k \nu^k \frac{d}{d\nu} (a_i \nu^i) := i z_k a_i \nu^{i+k-1}, \quad z_k \in Z(\mathcal{A}).$$

Here $Z(\mathcal{A})$ is the space of central elements. The induced Nijenhuis operator is

$$N(a_i \nu^i) := \frac{i}{i+k} z_k a_i \nu^{i+k}.$$

Example 6.4. Let $W\langle x, \partial_x \rangle$ be the Weyl algebra. Define a formal integral operator by, for the normal basis of the Weyl algebra,

$$\int dx \partial_x^i * x^j := \frac{1}{1+j} \partial_x^i * x^{j+1}, \quad i, j \geq 0.$$

Then the integral operator is a Rota-Baxter operator with weight zero (see [23]). We put $\Omega := i_{\partial_x}$. Then Ω is a strong Maurer-Cartan operator. Thus the composition map

$$N(u) := \int dx \Omega(u) = \int dx [\partial_x, u]$$

is a Nijenhuis operator on $W\langle x, \partial_x \rangle$. Since an arbitrary element u has the form of $u := k_{ij} \partial_x^i * x^{j(j \neq 0)} + k_i \partial_x^i + k$, we have $N(u) = k_{ij} \partial_x^i * x^{j(j \neq 0)}$. Thus N is a projection onto the space of elements of the form $k_{ij} \partial_x^i * x^{j(j \neq 0)}$.

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